The Survival and Extinction of Overconfident Agents

Steven D. Baker
sdbaker@cmu.edu

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Abstract

I study the survival of an irrationally overconfident agent, who competes with a rational agent in a stochastic growth economy subject to a short-sale constraint. I find that the constraint increases the overconfident agent's chance of survival, often providing an infinite relative improvement asymptotically. The magnitude of the effect increases with the level of overconfidence, and becomes stronger as the source of economic risk shifts from the growth process to direct innovations in the dividend process. As the overconfident agent loses his endowment, the constraint binds more often, altering the interest rate and his price of risk. A survival index is constructed, which links overconfidence with survival effects due to impatience and risk aversion.

1 Introduction

This paper studies the survival of an irrationally overconfident agent who competes with a rational agent. The overconfident agent is certain that a signal carries information about future economic growth, when in fact it does not. He is irrational in the sense that he never learns that the signal is meaningless. Both agents are otherwise fully rational Bayesian optimizers of expected intertemporal utility.

Trading takes place in continuous time in a pure exchange economy with a single consumption good. Agents, who have heterogeneous CRRA utility and impatience, optimize over an infinite horizon. The aggregate dividend process and spurious signal are exogenous. Prices, the interest rate, and other aspects of market behavior are determined endogenously via market clearing and the characteristics of the agents.

Survival is investigated in markets where short-sales are allowed, and in those where they are not. Traders often face impediments to short-selling, for a variety of reasons. Regulatory bodies such as the Federal Reserve and the SEC have at times restricted short-sales, and individual funds may be contractually prohibited from short-selling. In some cases it may be difficult to sell short even absent restrictions. My implementation of the short-sale constraint is based upon Gallmeyer and Hollifield [2008], which also provides a more detailed review of the topic. In the constrained setting, analysis is conducted for logarithmic utility only.

Of the many behavioral phenomena invoked as possible explanations for anomalies in financial markets, overconfidence has the advantage of being a widespread and well documented psychological trait, with possible evolutionary advantages explaining its apparent prevalence and persistence. For a survey of overconfidence in connection with finance, see for example Daniel and Titman [2000]. Overconfidence is also able to explain several well known financial phenomena at a stroke. Harrison and Kreps [1978] provide an important early study in a discrete setting without short-sales, where agents with heterogeneous beliefs equivalent to overconfidence are shown to speculate, creating a price bubble. Scheinkman and Xiong [2003] formulate the problem in continuous time, and argue that high trading volume and volatility may also be explained by overconfidence. Their work examines the effects of trading costs upon these phenomena, but as budget constraints and fully rational traders are absent, the survival of overconfident traders is not considered. My work builds upon that of Dumas et al. [2009], who include both fully rational and overconfident agents, with CRRA power utility and budget constraints. However, survival of overconfident agents is only examined under one set of parameter assumptions, with homogeneous preferences and impatience. In their parameterization, overconfident agents remain a significant fraction of the market for long periods (hundreds of years).

The survival of the overconfident agents is of critical importance to the validity of the overall theory, since otherwise it could be argued that they would quickly be forced from the markets, reducing their price impact and perhaps invalidating overconfidence as an explanation for observed speculative phenomena. I focus on the survival question under realistic and previously unexamined conditions. Important analysis of survival includes that of Kogan et al. [2006], who examine agents with heterogeneous beliefs and CRRA preferences who consume only in a terminal period. They conclude that agents with incorrect beliefs may dominate the market given appropriate preferences. More surprising, agents with negligible wealth can impact prices significantly. This effect vanishes when agents have intermediate consumption, as reported by Yan [2008]. However, Yan also concludes that disadvantages due to incorrect beliefs are easily overcome by the effects of heterogeneous preferences. He also introduces a "survival index", providing an elegant method of characterizing the asymptotic survival of heterogeneous agents. I extend the analysis of Yan [2008] to include stochastic growth and overconfidence, and generalize his survival index.

2 The Model

The model is identical to that of Dumas et al. [2009]. The exogenous aggregate output of the economy $\delta(t)$ follows a geometric Brownian motion with dynamics

$$\begin{aligned} d\delta(t) &= \delta(t)[f(t)dt + \sigma_\delta dZ_\delta(t)],\ \delta(0) > 0, \\ \Rightarrow \delta(t) &= \delta(0)\exp\{\int_0^t (f(s) - 1/2\sigma_\delta^2)ds + \sigma_\delta Z_\delta(t)\} \end{aligned} \tag{1}$$

where $Z_\delta(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ with $\sigma_\delta > 0$. f, the mean rate of economic growth, follows a mean reverting Ornstein-Uhlenbeck process:

$$\begin{aligned} df(t) &= -\zeta(f(t) - \bar{f})dt + \sigma_f dZ_f(t),\ \zeta > 0 \\ \Rightarrow f(t) &= e^{-\zeta t}[f(0) - \bar{f} + \sigma_f \int_0^t e^{\zeta s} dZ_f(s)] + \bar{f} \end{aligned} \tag{2}$$

In addition, there is a signal that follows

$$ds(t) = \sigma_s dZ_s(t), \tag{3}$$

where $Z_s(t)$ is a standard Brownian motion independent of $Z_\delta(t), Z_f(t)$. Assume there are two agents (investors), 1 and 2, who have different beliefs regarding future economic growth. Agent 1 correctly perceives s as independent of $Z_f(t)$, whereas agent 2 is irrationally "overconfident" that s is correlated with $Z_f(t)$ as follows:

$$ds(t) = \sigma_s \phi dZ_f(t) + \sigma_s \sqrt{1 - \phi^2} dZ_s(t). \tag{4}$$

Agent 2 always believes $\phi > 0$ and constant. Neither agent observes f, neither are the innovations directly observable, but all other aspects of the economic model are assumed common knowledge unless otherwise stated. From [Liptser and Shiryaev, 1977, Thm. 12.7, p.33], based on their respective beliefs, the agents estimate the growth rate as

$$df_1(t) = -\zeta(f_1(t) - \bar{f})dt + \frac{\varphi_1}{\sigma_\delta^2}\left(\frac{d\delta(t)}{\delta(t)} - f_1(t)dt\right) \tag{5}$$

$$\Rightarrow f_1(t) = e^{-\zeta t}[f_1(0) - \bar{f} + \frac{\varphi_1}{\sigma_\delta}\int_0^t e^{\zeta s} dW_\delta^1(s)] + \bar{f}, \tag{6}$$

$$df_2(t) = -\zeta(f_2(t) - \bar{f})dt + \frac{\varphi_2}{\sigma_\delta^2}\left(\frac{d\delta(t)}{\delta(t)} - f_2(t)dt\right) + \frac{\phi\sigma_f}{\sigma_s}ds \tag{7}$$

$$\Rightarrow f_2(t) = e^{-\zeta t}[f_2(0) - \bar{f} + \frac{\varphi_2}{\sigma_\delta}\int_0^t e^{\zeta s} dW_\delta^2(s) + \phi\sigma_f\int_0^t e^{\zeta s} dZ_s(s)] + \bar{f} \tag{8}$$

Above, $dW_\delta^1(t) = \frac{1}{\sigma_\delta}(\frac{d\delta(t)}{\delta(t)} - f_1(t)dt)$ and $dW_\delta^2(t) = \frac{1}{\sigma_\delta}(\frac{d\delta(t)}{\delta(t)} - f_2(t)dt)$ are standard one dimensional Brownian motions with respect to the beliefs of agents 1 and 2, respectively. The estimated steady state errors of the growth rate approximations are

$$\varphi_1 = \sigma_\delta^2\left(\sqrt{\zeta^2 + \frac{\sigma_f^2}{\sigma_\delta^2}} - \zeta\right) \tag{9}$$

$$\varphi_2 = \sigma_\delta^2\left(\sqrt{\zeta^2 + (1-\phi^2)\frac{\sigma_f^2}{\sigma_\delta^2}} - \zeta\right) \tag{10}$$

Although the real economy is described in terms of three independent Brownian motions, only two random processes are actually observed by either agent. It is convenient to conduct equilibrium analysis in the space of observable random processes. See Riedel [2001] for a theoretical justification of this approach , and Dumas et al. [2009] for relevant derivations. Under agent 1's beliefs, the exogenous aggregate output of the economy and the signal s follow

$$\begin{aligned} d\delta(t) &= \delta(t)[f_1(t)dt + \sigma_\delta dW_\delta^1(t)],\ \delta(0) > 0 \\ df_1(t) &= -\zeta(f_1(t) - \bar{f})dt + \frac{\varphi_1}{\sigma_f}dW_\delta^1(t) \\ ds(t) &= \sigma_s dW_s^1(t) \end{aligned} \tag{11}$$

where W_s^1 is a one dimensional Brownian motion independent of W_δ^1.

3 Equilibrium

Assume there are three linearly independent assets such that markets are dynamically complete (in the sense that the attainable consumption streams span the space of observable random processes). In particular, I assume that shares of aggregate dividends are in net supply 1 and tradeable as a stock, whereas other assets necessary for market completeness are in net supply 0. Each agent i is endowed at time $t = 0$ with a fraction of the dividend stream θ_i, and has time additive CRRA preferences given by

$$u_i(c_i(t)) = \frac{c_i(t)^{1-\gamma_i}}{1-\gamma_i},\ \gamma_i > 0, \tag{12}$$

with $\gamma_i = 1$ corresponding to $u_i(c_i(t)) = \log c_i(t)$. The first derivative of the utility function and its inverse are given by

$$u_i'(c_i(t)) = c_i(t)^{-\gamma_i} \tag{13}$$
$$J(v) = v^{-1/\gamma_i} \tag{14}$$

As shown in Karatzas et al. [1990], an agent's optimal consumption can be written as $c_i(t) = J_i(e^{\rho_i t} y_i \xi_i(t))$, where ξ_i is the agent's state price density, $\rho_i > 0$ is a constant reflecting impatience and y_i is a constant that satisfies

$$E^i[\int_0^\infty \xi_i(t) J_i(e^{\rho_i t} y_i \xi_i(t))dt] = E^i[\int_0^\infty \xi_i(t)\theta_i \delta(t)dt] = x_i. \tag{15}$$

Above, E^i is expectation with respect to agent i's beliefs and x_i is initial wealth. I will now construct a utility function for the representative agent following the approach of Basak [2005]. Let

$$U(\delta(t); \lambda_{1,2}(t)) = \max_{c_1(t)+c_2(t)\le\delta(t)} \lambda_1(t)e^{-\rho_1 t}u_1(c_1(t)) + \lambda_2(t)e^{-\rho_2 t}u_2(c_2(t)) \tag{16}$$

or equivalently we can normalize the maximization problem by $\lambda_1(t)$ and write

$$u(\delta(t); \lambda(t)) = \max_{c_1(t)+c_2(t)\le\delta(t)} e^{-\rho_1 t}u_1(c_1(t)) + \lambda(t)e^{-\rho_2 t}u_2(c_2(t)) \tag{17}$$

where $\lambda(t) = \frac{\lambda_2(t)}{\lambda_1(t)}$. Dropping the time parameter for notational simplicity, we have the following Lagrangian and its derivatives

$$\begin{aligned} L(c_1, c_2, \delta; \lambda) &= e^{-\rho_1 t}u_1(c_1) + \lambda e^{-\rho_2 t}u_2(c_2) - \mu(c_1 + c_2 - \delta) \\ \mu &= e^{-\rho_1 t}u_1'(c_1) \\ \mu &= \lambda e^{-\rho_2 t}u_2'(c_2) \\ \Rightarrow \lambda &= \frac{e^{-\rho_1 t}u_1'(c_1)}{e^{-\rho_2 t}u_2'(c_2)} \end{aligned} \tag{18}$$

Application of the envelope theorem yields

$$\begin{aligned} u'(\delta; \lambda) &= \frac{\partial L}{\partial \delta} \\ &= e^{-\rho_1 t}u_1'(c_1)\frac{\partial c_1}{\partial \delta} + \lambda e^{-\rho_2 t}u_2'(c_2)\frac{\partial c_2}{\partial \delta} + \mu \\ &= \mu = e^{-\rho_1 t}u_1'(c_1) = \lambda e^{-\rho_2 t}u_2'(c_2) \end{aligned} \tag{19}$$

To summarize, we have

$$\lambda(t) = \frac{e^{-\rho_1 t}u_1'(c_1(t))}{e^{-\rho_2 t}u_2'(c_2(t))} = \frac{y_1\xi_1(t)}{y_2\xi_2(t)} \tag{20}$$

Since it is the ratio $\frac{y_1}{y_2}$ that matters in determining $\lambda(t)$ and the resulting state price densities, I can choose

$$y_1 = u'(\delta(0)) \Rightarrow \xi_1(t) = \frac{u'(\delta(t))}{u'(\delta(0))}. \tag{21}$$

Consider time $t = 0$. Then $\xi_1(0) = 1$ and

$$\begin{aligned} \lambda(0) &= \frac{u'(\delta(0))}{y_2\xi_2(0)} \\ \Rightarrow y_2 &= \frac{u'(\delta(0))}{\lambda(0)\xi_2(0)} \\ \Rightarrow \xi_2(t) &= \frac{\xi_2(0)\lambda(0)u'(\delta(t))}{\lambda(t)u'(\delta(0))}. \end{aligned} \tag{22}$$

Setting the initial value $\xi_2(0) = 1$ satisfies the above equation to give

$$\xi_2(t) = \frac{\lambda(0)}{\lambda(t)}\xi_1(t) \tag{23}$$

It remains to determine the process $\lambda(t)$. Define the difference of opinion between agents as

$$g(t) = \frac{f_1(t) - f_2(t)}{\sigma_\delta} \tag{24}$$

The relationship $W_\delta^2(t) = W_\delta^1(t) + \int_0^t g(s)ds$ suggests an application of Girsanov's theorem to construct a change of measure between the probability spaces $\mathbb{P}^1, \mathbb{P}^2$ of the two agents. Suppose

$$\begin{aligned} \eta(t) = \frac{\lambda(t)}{\lambda(0)} &= \exp\{-\int_0^t g(s)dW_\delta^1(t) - 1/2\int_0^t g^2(s)ds\} \\ d\lambda(t) &= -\lambda(t)g(s)dW_\delta^1(t). \end{aligned} \tag{25}$$

Then $\eta = \eta(\infty)$ implements a change of measure from $\mathbb{P}^2$ to $\mathbb{P}^1$, such that for any event A

$$\begin{aligned} E_t^2[\xi_2(t)\mathbf{1}_A] &= E_t^1[\eta\xi_2(t)\mathbf{1}_A] = \eta(t)\xi_2(t)\mathbb{P}^1(A) \\ &= E_t^1[\xi_1(t)\mathbf{1}_A] = \xi_1(t)\mathbb{P}^1(A) \\ \Rightarrow \xi_2(t) &= \frac{1}{\eta(t)}\xi_1(t) = \frac{\lambda(0)}{\lambda(t)}\xi_1(t) \end{aligned} \tag{26}$$

which establishes that the process for $\lambda(t)$ is as posited. $\lambda(0)$ is selected to satisfy the budget constraint (15). Although the preceding discussion characterizes equilibrium, it does not guarantee existence or uniqueness. Indeed, this is difficult to show in the general case; Karatzas et al. [1990] and Yan [2008] provide proofs under certain conditions. Henceforth I assume the existence of a unique equilibrium.

It is possible to characterize $\lambda(t)$ in a different way, using the market price of risk (i.e., the instantaneous Sharpe ratio). This is defined as

$$\kappa_i(t) = \frac{f_i(t) - r(t)}{\sigma_\delta} \tag{27}$$

where $r(t)$ is the instantaneous risk free interest rate (excluding the term for impatience). Then the evolution of lambda is governed by

$$\begin{aligned} d\lambda(t) &= -\lambda(t)\sigma_\lambda(t)dW_\delta^1(t), \\ \sigma_\lambda(t) &= \kappa_1(t) - \kappa_2(t) \end{aligned} \tag{28}$$

Likewise, the change of measure between agent i's perceptions and the true economy is given by

$$\begin{aligned} d\lambda_i(t) &= -\lambda_i(t)h_i(t)dZ_\delta(t) \\ h_i(t) &= \kappa(t) - \kappa_i(t),\ \kappa(t) = \frac{f(t) - r(t)}{\sigma_\delta} \\ \Rightarrow \lambda_i(t) &= \lambda_i(0)\exp\{-1/2\int_0^t h_i(s)^2 ds - \int_0^t h_i(s)dZ_\delta(s)\} \end{aligned} \tag{29}$$

If at present the two definitions of $\lambda(t)$ seem to contribute nothing more than a surplus of notation, their usefulness will become apparent when short-sale constraints are introduced, as they will not remain equivalent.

4 Survival

Survival may be defined either in terms of an agent's asymptotic share of wealth, or his asymptotic share of consumption. Dumas et al. [2009] argues that consumption share is the more informative statistic in the context of this model. Therefore we have the following definition.

Definition 1. *An agent's share of consumption is given by*

$$\omega_i(t) = \frac{c_i(t)}{\delta(t)} \tag{30}$$

Agent i becomes "extinct" if

$$\lim_{t\to\infty} \omega_i(t) = 0, \ a.s., \tag{31}$$

and "survives" if this does not occur. He "dominates" the market if

$$\lim_{t\to\infty} \omega_i(t) = 1, \ a.s. \tag{32}$$

In later sections I assume that all agents have log utility. In this case share of wealth and share of consumption are equivalent, so results for survival may be compared directly with prior work in terms of either wealth or consumption.

Yan [2008] introduces the concept of a "survival index" that determines which agent survives in the limit. As in this paper, his agents may have heterogeneous patience, risk aversion, and beliefs. However, the focus of Yan's analysis is an economy with constant growth and constant errors in the agents' beliefs, with some analysis of learning given heterogeneous priors. As far as I am aware, the extension of the survival index to overconfident agents in a stochastic growth economy is novel.

Survival is analyzed with respect to the true economy governed by $Z_\delta(t)$, such that $dW_i(t) = dZ_\delta(t) + h_i(t)$. In the absence of a short-sale constraint, $h_i(t) = \frac{f(t)-f_i(t)}{\sigma_\delta}$. Note that both the rational and the overconfident agent estimate growth correctly on average, i.e., $E[h_i(t)] = 0$. [1] The definition and proposition below help to characterize survival in this economy.

Definition 2. *The survival index for agent i at time t is*

$$I_i(t) = \frac{1}{2t}\int_0^t h_i(s)^2 ds + \rho_i + \frac{\gamma_i}{t}\int_0^t f(s) - 1/2\sigma_\delta^2 ds. \tag{33}$$

The long-run survival index for agent i is

$$\bar{I}_i = \lim_{t\to\infty} I_i = \psi_i + \rho_i + \gamma_i(\bar{f} - 1/2\sigma_\delta^2) \tag{34}$$

where ψ_i is half the steady-state error in agent i's estimate of economic growth normalized by volatility σ_δ. For agents with varying degrees of overconfidence (whose unconditional estimates are unbiased), this reduces to half the steady state variance of $h_i(t)$ or, equivalently, the volatility of $\lambda_i(t)$. [2] *I refer to ψ_i as the belief penalty.*

[1] In the case of agent 1, this results from the construction of $f_1(t)$, which follows Theorem 12.1 in Liptser and Shiryaev [1977]. While the condition $E[f_2(t)] = \bar{f}$ is intuitively satisfied due to the symmetry of the spurious signal, the fact that $f_2(t)$ is misspecified relative to the true model makes a formal proof more difficult. However, the property has been confirmed via simulation.

[2] Once again, convergence of $Var(h_1)$ follows from Liptser and Shiryaev [1977] and the resulting process $\varphi_1(t)$, whereas convergence and estimation of $Var(h_2)$ has so far only been established numerically.

Proposition 1. *The agent with the smallest long-run survival index survives asymptotically.*

Proof. The proof is adapted from that of Proposition 2 in Yan [2008]. Although it is given with the case of one rational agent and one overconfident agent in mind, the result is readily generalized to the case of N agents with more general h_i; only the definition of $\bar{I}_i$ would change. From equation 18,

$$\begin{aligned} \frac{\lambda_2(t)}{\lambda_1(t)} &= \frac{e^{-\rho_1 t} c_1(t)^{-\gamma_1}}{e^{-\rho_2 t} c_2(t)^{-\gamma_2}} \\ \Rightarrow \frac{c_2(t)^{\gamma_2}}{c_1(t)^{\gamma_1}} &= e^{(\rho_1-\rho_2)t} \frac{\lambda_2(t)}{\lambda_1(t)} \\ \Rightarrow \frac{\omega_2^{\gamma_2}(t)}{\omega_1^{\gamma_1}(t)} &= e^{(\rho_1-\rho_2)t} \frac{\lambda_2(t)}{\lambda_1(t)} \delta(t)^{\gamma_1-\gamma_2} \end{aligned} \tag{35}$$

Substituting the solutions for $\lambda_i(t)$ and $\delta(t)$ leads to

$$\begin{aligned} \frac{\omega_2^{\gamma_2}(t)}{\omega_1^{\gamma_1}(t)} &= \frac{\lambda_2(0)}{\lambda_1(0)} \delta(0)^{\gamma_1-\gamma_2} \exp\{(\rho_1-\rho_2)t + 1/2 \int_0^t (h_1(s)^2 - h_2(s)^2) ds + \int_0^t (h_1(s) - h_2(s)) dZ_\delta(s) \\ &\quad + (\gamma_1-\gamma_2)[\int_0^t (f(s) - 1/2\sigma_\delta^2) ds + \sigma_\delta Z_\delta(t)]\} \\ &= \frac{\lambda_2(0)}{\lambda_1(0)} \delta(0)^{\gamma_1-\gamma_2} \exp\{(I_1 - I_2)t + (\int_0^t (h_1(s) - h_2(s)) dZ_\delta(s) + (\gamma_1-\gamma_2)\sigma_\delta Z_\delta(t))\} \\ &= \frac{\lambda_2(0)}{\lambda_1(0)} \delta(0)^{\gamma_1-\gamma_2} \exp\{[(I_1 - I_2) + \frac{1}{t}(\int_0^t (h_1(s) - h_2(s)) dZ_\delta(s) + (\gamma_1-\gamma_2)\sigma_\delta Z_\delta(t))]t\} \end{aligned} \tag{36}$$

Recall that the a law of large numbers implies $\lim_{t\to\infty} \frac{Z_\delta(t)}{t} = 0$, *a.s.* (see for example [Karatzas and Shreve, 1991, p. 104]). If we assume a bound $|h_i(t)| < \bar{h}$, $\forall i, t$ on the error agents can reasonably make in estimating the growth rate, then

$$\lim_{t\to\infty} \frac{1}{t} | \int_0^t (h_1(s) - h_2(s)) dZ_\delta(s) + (\gamma_1-\gamma_2)\sigma_\delta Z_\delta(t)| \leq (2\bar{h} + |(\gamma_1-\gamma_2)\sigma_\delta|) |\frac{Z_\delta(t)}{t}| = 0, \ a.s. \tag{37}$$

It follows that

$$\lim_{t\to\infty} (I_1 - I_2) + \frac{1}{t}(\int_0^t (h_1(s) - h_2(s)) dZ_\delta(s) + (\gamma_1-\gamma_2)\sigma_\delta Z_\delta(t)) = \bar{I}_1 - \bar{I}_2, \ a.s. \tag{38}$$

Therefore if $\bar{I}_1 > \bar{I}_2$,

$$\lim_{t\to\infty} \frac{\omega_2^{\gamma_2}(t)}{\omega_1^{\gamma_1}(t)} = \lim_{t\to\infty} \frac{\lambda_2(0)}{\lambda_1(0)} \delta(0)^{\gamma_1-\gamma_2} \exp\{(\bar{I}_1 - \bar{I}_2)t\} = \infty, \ a.s. \tag{39}$$

Recalling that $0 \leq \omega_i(t) \leq 1$, the above implies

$$\lim_{t\to\infty} \omega_1(t) = 0, \ a.s. \tag{40}$$

The equivalent result applies for $\bar{I}_1 < \bar{I}_2$, implying that the agent with the smallest limiting survival index survives [3], whereas the other agent becomes extinct. If $\bar{I}_1 = \bar{I}_2$, then both agents survive in the limit. □

[3] In fact, since there are only two agents, the survivor dominates

4.1 Discussion

The survival index provides a convenient way to evaluate the asymptotic survival of heterogeneous agents. The interaction of heterogeneous impatience and risk aversion with incorrect beliefs is characterized by Yan [2008], and the basic characterization remains unchanged. However, the heterogeneous beliefs arising due to overconfidence and stochastic growth lead to different finite-horizon dynamics. In addition, the relative magnitude of the belief effects changes with the introduction of stochastic growth, as does the definition of rational beliefs: no agent perfectly estimates growth asymptotically if $\sigma_f > 0$. Accordingly, the following proposition summarizes optimal beliefs in this economy.

Proposition 2. *The optimal (minimum mean-square error) estimate of the growth rate $f(t)$ is given by $f_1(t)$, with asymptotic error in beliefs characterized by*

$$\psi_1 = \frac{\varphi_1}{2\sigma_\delta^2} = \frac{1}{2}\left(\sqrt{\zeta^2 + \frac{\sigma_f^2}{\sigma_\delta^2}} - \zeta\right), \tag{41}$$

where φ_1 is the steady-state error of the optimal filter. Assume agents have identical impatience and risk aversion. Then agent 1 survives. If in addition agents have identical initial growth estimate $f_i(0) = E[f(0)]$, then the optimal filter $f_1(t)$ is unique, and agent 1 dominates.

Proof. Survival follows from Theorem 12.1 in Liptser and Shiryaev [1977], which establishes optimality of $f_1(t)$, and Proposition 1, which shows the connection between minimum error growth estimates and survival. Dominance follows from Theorem 12.3 in Liptser and Shiryaev [1977], which guarantees uniqueness of the optimal filter up to initial conditions. □

Although I focus on competition between the rational agent 1 (using the optimal growth estimate) and an overconfident agent (who uses a misspecified filter), the proposition above establishes survival and dominance of agent 1 versus *any* other investor, including (for example) an investor who simply estimates growth as its long run mean $\bar{f}$. Also note that the initial growth estimate $f_i(0)$ - the source of heterogeneous beliefs considered in Yan [2008] and Gallmeyer and Hollifield [2008] - is of secondary importance here. In fact, agent 1 will dominate any agent who does not estimate growth in the form of $f_1(t)$; the assumption of identical initial conditions merely contains the case where another agent uses the optimal filter up to an irrational estimate of the initial condition. (Knowledge of the distribution of $f(0)$ is a necessary assumption for construction of the optimal filter.)

5 Survival Under Short-sale Constraint

To simplify implementation of the short-sale constraint and focus upon its interactions with overconfidence and stochastic growth, some standing assumptions are made. In particular, all agents are assumed to have logarithmic utility ($\gamma_i = 1$). Agents also have uniform impatience $\rho_i = 0.1$, matching the parameter from Dumas et al. [2009], although heterogeneous impatience is explored at the end of this section. These assumptions make results easier to interpret, as several quantities become available in closed form.

Suppose we characterize the agents according to their current relative beliefs, calling them o (optimistic) and p (pessimistic), $f_o(t) \geq f_p(t)$. Define the difference of opinion $\hat{g}(t) = \frac{f_o(t) - f_p(t)}{\sigma_\delta}$. Then the representative agent has the utility maximization problem

$$\hat{u}(\delta(t); \hat{\lambda}(t)) = \max_{c_o(t)+c_p(t)\leq\delta(t)} e^{-\rho_o t} u_o(c_o(t)) + \hat{\lambda}(t) e^{-\rho_p t} u_p(c_p(t)) \tag{42}$$

Gallmeyer and Hollifield [2008] show that, under these assumptions,

$$d\hat{\lambda}(t) = -\hat{\lambda}(t)\sigma_{\hat{\lambda}}(t)dW_{\delta}^{o},\ \sigma_{\hat{\lambda}}(t) = \kappa_o(t) - \kappa_p(t), \tag{43}$$

and κ_i is given by the table below

Constraint Binds	$\kappa_o(t)$	$\kappa_p(t)$
Yes	$(1+\hat{\lambda}(t))\sigma_\delta$	0
No	$\sigma_\delta + \frac{\hat{\lambda}(t)\hat{g}(t)}{1+\hat{\lambda}(t)}$	$\sigma_\delta - \frac{\hat{g}(t)}{1+\hat{\lambda}(t)}$

The constraint binds when $\kappa_p(t)$ would otherwise become negative. Note that this is more likely to occur when agent p's consumption share is small.

In the previously described economy with one rational and one overconfident agent, numbers 1 and 2, respectively, the optimistic and pessimistic agent fluctuates with the sign of $g(t)$. Suppose $f_1(t) \geq f_2(t)$. Then the representative agent's utility maximization problem corresponds to equation (43) with $\hat{\lambda}(t) = \frac{\lambda_2(t)}{\lambda_1(t)} = \lambda(t)$, $o = 1$, $p = 2$. Similarly, if $f_1(t) \leq f_2(t)$, then $\hat{\lambda}(t) = \frac{\lambda_1(t)}{\lambda_2(t)} = \frac{1}{\lambda(t)}$, $o = 2$, $p = 1$. When $f_1(t) = f_2(t)$, the two problems are equivalent, as the constraint will not bind either agent. Prices of risk and the interest rate (also adapted from Gallmeyer and Hollifield [2008]) are in Table 1.

Constraint Binds	$\kappa_1(t)$	$\kappa_2(t)$	r(t)
Agent 1	0	$\left(1+\frac{1}{\lambda(t)}\right)\sigma_\delta$	$\rho + f_2(t) - \left(1+\frac{1}{\lambda(t)}\right)\sigma_\delta^2$
Agent 2	$(1+\lambda(t))\sigma_\delta$	0	$\rho + f_1(t) - (1+\lambda(t))\sigma_\delta^2$
Neither agent	$\sigma_\delta + \frac{\lambda(t)g(t)}{1+\lambda(t)}$	$\sigma_\delta - \frac{g(t)}{1+\lambda(t)}$	$\rho + \frac{f_1(t)+\lambda(t)f_2(t)}{\lambda(t)+1} - \sigma_\delta^2$

Table 1: Prices of risk and interest rate under short-sale constraint

Note that the last row of Table 1 is identical to the case of the unconstrained economy.

Because the hypothetical omniscient agent (who observes $f(t)$) does not participate in the economy, he has no impact upon the interest rate. Hence his price of risk is simply $\kappa(t) = \max(\frac{f(t)-r(t)}{\sigma_\delta}, 0)$. Recalling that $h_i(t) = \kappa(t) - \kappa_i(t)$ (which *does not* reduce to the difference of growth rates when short-sales are restricted), we have everything necessary to interpret the survival index under the constraint.

Definition 3. *When short-sales are disallowed, the survival index for agent i at time t remains*

$$I_i(t) = \frac{1}{2t}\int_0^t h_i(s)^2 ds + \rho_i + \frac{\gamma_i}{t}\int_0^t f(s) - 1/2\sigma_\delta^2 ds. \tag{44}$$

The long-run survival index for agent i is

$$\bar{I}_i = \lim_{t\to\infty} I_i = \psi_i(\bar{\lambda}) + \rho_i + \gamma_i(\bar{f} - 1/2\sigma_\delta^2) \tag{45}$$

where $\psi_i(\bar{\lambda})$ is half the steady-state error of agent i's estimated price of risk versus the true price of risk. $\bar{\lambda}$ signifies the asymptotic distribution of the weighting factor $\lambda(t)$.

Thus the survival index may in fact depend on whether the agent survives! This conundrum will be examined in the context of an experimental simulation. To simplify notation, the $\bar{\lambda}$ argument is omitted in future discussion.

5.1 Simulation and Baseline Parameters

Monte Carlo simulation is used to evaluate survival in a variety of scenarios. Antithetic variates were found to reduce error somewhat, and are used in all experiments. Unless otherwise noted, results are generated by simulating the economy 400,000 times. For terminal period $T = 100$, integrals are approximated with time step $dt = 0.02$. For $T = 1000$, $dt = 0.05$. All times are in years. Further details accompany each figure.

Name	**Symbol**	**Value**
Parameters from Dumas et al. [2009]		
Mean growth rate of aggregate endowment	$\bar{f}$	0.015
Volatility of endowment growth rate	σ_f	0.03
Volatility of aggregate endowment	σ_δ	0.13
Mean reversion parameter	ζ	0.2
Agent 2's correlation between signal and growth rate	ϕ	0.95
Agent's initial shares of endowment	$\theta_1 = \theta_2$	0.5
Initial aggregate dividend	$\delta(0)$	1
Initial growth expectations	$f(0) = f_1(0) = f_2(0)$	$\bar{f}$
Parameters from Yan [2008]		
Constant growth rate	$\tilde{f}$	0.018
Volatility of aggregate endowment	$\tilde{\sigma}_\delta$	0.032

Table 2: Baseline parameter values and state variables

Table 2 gives the baseline parameters used to assess survival in this paper. They match those used by Dumas et al. [2009], which were in turn based on estimates from Brennan and Xia [2001]. For comparison, I have also included the core economic parameters from the constant growth economy of Yan [2008]. His constant growth rate $\tilde{f}$ is most equivalent to my $\bar{f}$, to which my experiments are generally insensitive. $\tilde{\sigma}_\delta$ may be compared to σ_δ when considering instantaneous risk, but this comparison fails in the context of the forecast risk that applies over longer periods. Section 5.3 examines the effect of reducing (increasing) σ_δ while increasing (reducing) risk from other sources.

5.2 Direct Effects of Overconfidence

What level of overconfidence is "reasonable" is open to speculation. The relatively large baseline value $\phi = 0.95$ was chosen primarily for consistency with prior work, and because it is large enough to provide a convincing "worst case scenario": if very overconfident agents survive for long periods, then clearly moderately overconfident agents will! But how important is the level of overconfidence when short-sales are disallowed?

It turns out to be less important. Figure 1 shows how the consumption share of the irrational agent evolves over time. The effect of the short-sale constraint can be dramatic. Under the baseline parameters ($\phi = 0.95$), the unconstrained overconfident agent is nearly extinct after 300 years, but still consumes around 5% of the dividend under the constraint. The impact of the constraint declines as overconfidence is reduced to $\phi = 0.5$, but it is still significant, offering a survival advantage through the terminal period. When agent 2 is only slightly overconfident his unconstrained wealth share is nearly indistinguishable from his constrained wealth share.

The accompanying bar chart of belief penalties (ψ) provides intuition for the results. The leftmost bars correspond to the rational agent, who has $\phi = 0$ but still has $\psi > 0$, because perfect estimation of the

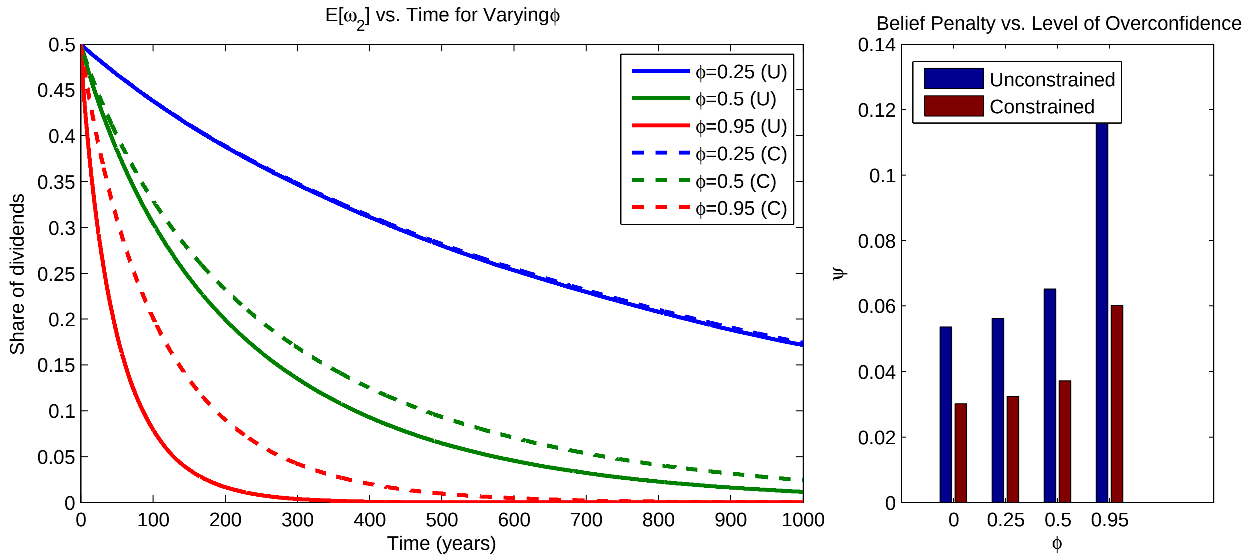

Figure 1: Impact of overconfidence parameter ϕ.
Maximum standard errors over the interval and all parameter combinations are as follows: $E[\omega_2] : 4.23 \times 10^{-4}$, $\psi : 5.49 \times 10^{-4}$.

growth rate is impossible given observable information. As all other components of the survival index are assumed identical for both agents, the agent with the smallest ψ survives. Further, because the wealth ratio is exponential in the survival index, it is the *absolute* difference $\psi_1 - \psi_2$ that determines the rate of agent 2's extinction, not the relative magnitude of the belief penalties. When modification of the baseline parameters is restricted to ϕ only, introduction of the short-sale constraint reduces a given belief penalty approximately linearly, by around 50%. Hence the absolute reduction in ψ is largest when the unconstrained ψ is large, e.g., for $\phi = 0.95$, when agent 2 is very overconfident. When $\psi_1 - \psi_2$ is small to begin with ($\phi = 0.25$), the constraint causes a very small absolute reduction in the penalty difference, and $E[\omega_2]$ is almost unaffected.

It is worth noting that imposing the short-sale constraint provides a *relative* increase in agent 2's consumption that grows over time. Figure 2 illustrates this point: the ratio of expected consumptions converges to zero with time, indicating that the overconfident agent's consumption is infinitely larger with the constraint than without it. However, when the benefit of the constraint is slight, the ratio may remain close to one for long periods.

	Unconstrained	**Constrained**
$E[u_1]$	-5.27	-5.80
$E[u_2]$	-11.90	-8.43

Table 3: Expected Discounted Utility through $T = 100$

To further illustrate who benefits from imposition of the constraint, Table 3 presents the cumulative expected discounted utility for each agent for the first 100 years under the baseline parameters. [4] Although his attempts to trade are more often frustrated by the constraint than those of agent 1, agent 2 has significantly higher expected utility under the constraint. Therefore if agent 2 could be "convinced" of his irrationality (although he remained powerless to alter his behavior), he would prefer to trade in a market with the short-sale constraint than without it.

[4]The estimate was computed using 100,000 samples with $dt = 0.02$. Because the discount factor is substantially larger than the growth rate, contributions to cumulative utility for $t > 100$ are slight.

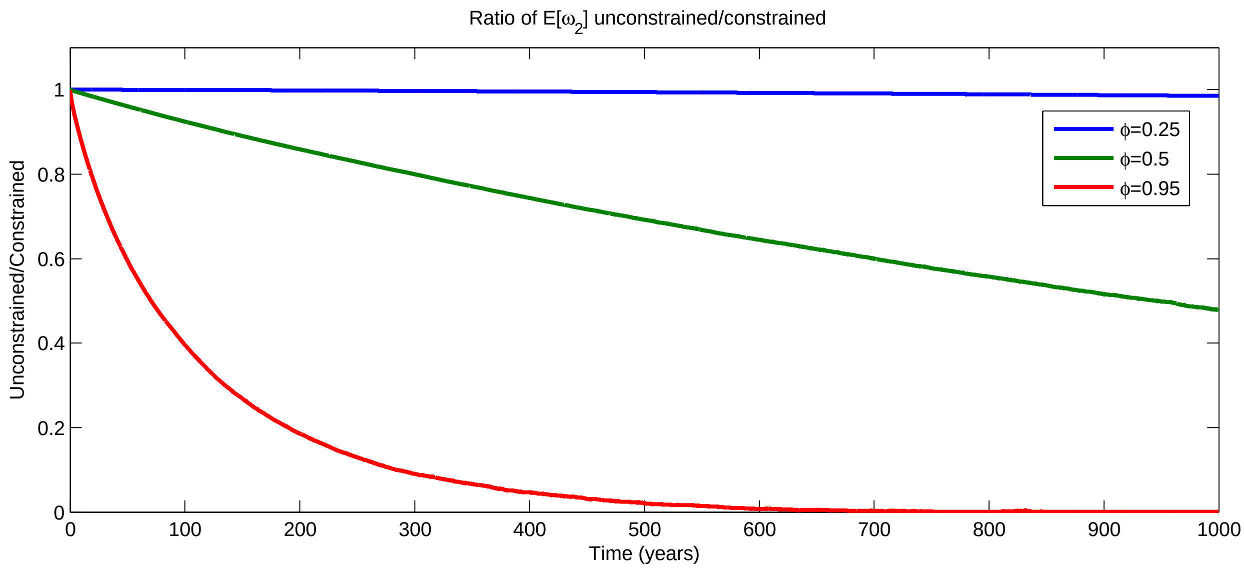

Figure 2: Relative increase in $E[\omega_2]$ due to short-sale constraint.
Maximum standard error over the interval and all parameter combinations is 4.23×10^{-4}.

5.3 Sensitivity to Economic Parameters

Reasonable economic parameters are more precisely determined than the level of overconfidence. However it is important to evaluate the impact upon survival of modest changes to economic fundamentals. I attempt this in a controlled fashion by simultaneously adjusting growth volatility σ_f and dividend volatility σ_δ. Using equation (46) below, we can increase or decrease σ_f while holding ψ_1 constant. Hence any impact upon the survival of agent 2 is due entirely to changes in *his* survival index; the parameter changes leave the index of agent 1 unaltered.

$$\sigma_\delta = \frac{\sigma_f}{\sqrt{(2\psi_1 + \zeta)^2 - \zeta^2}} \tag{46}$$

An interesting side effect of this normalization is that the overconfident agent's *perceived* belief penalty (equation (47)) also remains unchanged.

$$\hat{\psi}_2 = \frac{1}{2}\left(\sqrt{\zeta^2 + (1-\phi^2)\frac{\sigma_f^2}{\sigma_\delta^2}} - \zeta\right) \tag{47}$$

This is important, as an examination of agent 2's growth rate estimate $f_2(t)$ reveals that his deficiencies stem from two sources. First, he underestimates the error of his estimate, which he perceives as $\varphi_2 < \varphi_1$. Because of this foolishness, agent 2 adjusts his growth estimate too little in response to observed changes to the dividend $\delta(t)$. It is φ_2 that governs the overconfident agent's perceived penalty $\hat{\psi}_2$. Since $\hat{\psi}_2$ remains fixed, the aforementioned parameter changes affect survival only through agent 2's second source of error: the pointless adjustments to his growth estimate in response to the signal. These adjustments are amplified by σ_f.

In Figure 3, σ_f is adjusted plus and minus 50% from the baseline parameter, while σ_δ is scaled to leave ψ_1 fixed, as described. Three interesting results emerge from this experiment. In the unconstrained case, it appears that leaving the overconfident agent's perceived belief penalty unchanged also leaves his *actual* penalty unchanged. Whether this relationship holds when other parameter combinations are adjusted in a similar way has yet to be investigated.

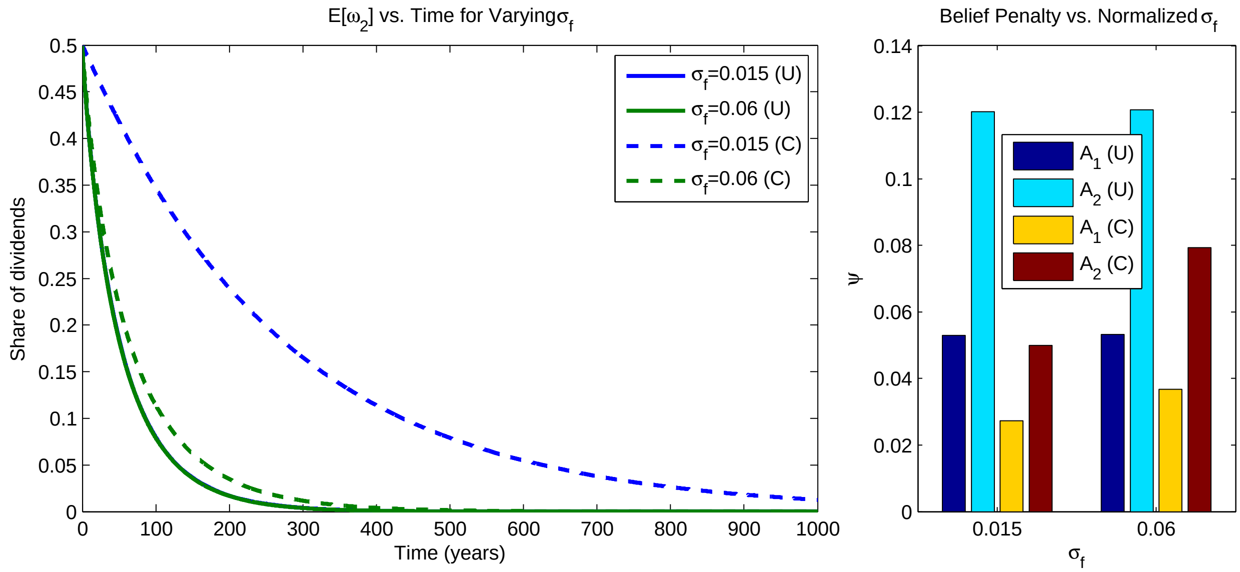

Figure 3: Normalized adjustment of σ_f.

Maximum standard errors over the interval and all parameter combinations are as follows: $E[\omega_2] : 4.24 \times 10^{-4}$, $\psi : 5.52 \times 10^{-4}$.

Matters are different when the short-sale constraint is imposed: the impact of the constraint is much more significant when growth volatility is low (but dividend volatility is high). Examination of the ψ values in the bar chart suggests that the constraint has a big effect on both agents when the dividend is noisy but the growth rate is not. The reason why this benefits agent 2 disproportionately is the same as discussed for varying ϕ: a big proportional reduction in ψ_1 and ψ_2 leads to a big absolute reduction in $\psi_1 - \psi_2$.

The third point emerging from Figure 3 relates to the direct linkage between the short-sale constraint and the consumption share: it is apparent from equation (43) that the short-sale constraint is more likely to bind an agent with a small consumption share. However in this experiment $E[\omega_2]$ would decline at the same rate in the unconstrained case under *either* set of parameters. Since the effect of the constraint is not felt identically, we can dispense with the idea that the constraint's impact is merely more significant when the overconfident agent's decline is more severe.

5.4 The Belief Penalty and Consumption Share

The top plot in Figure 4 explores the interaction between the short-sale constraint, consumption share, and belief penalty in more detail. Under the baseline parameterization, $f(t)$, $f_1(t)$, and $f_2(t)$ are simulated until they achieve a steady state, in which each agent's estimation of $f(t)$ is (on average) unchanged from one period to the next. Consumption share is then adjusted, from dominance of agent 2 at the left, to dominance of agent 1 at the right. As an agent loses wealth, the constraint binds him more frequently. This is because his reduced consumption share leads to a smaller impact upon the equilibrium interest rate: the interest rate declines less in response to a poor agent's low growth estimate, since he owns a small share of the endowment. The effect upon the belief penalties is more complicated. ψ_i captures the difference between agent i's perceived market price of risk and that of an omniscient agent trading under the same economy subject to the same interest rate. Interestingly, this difference is minimized when the two agents have the same consumption share.

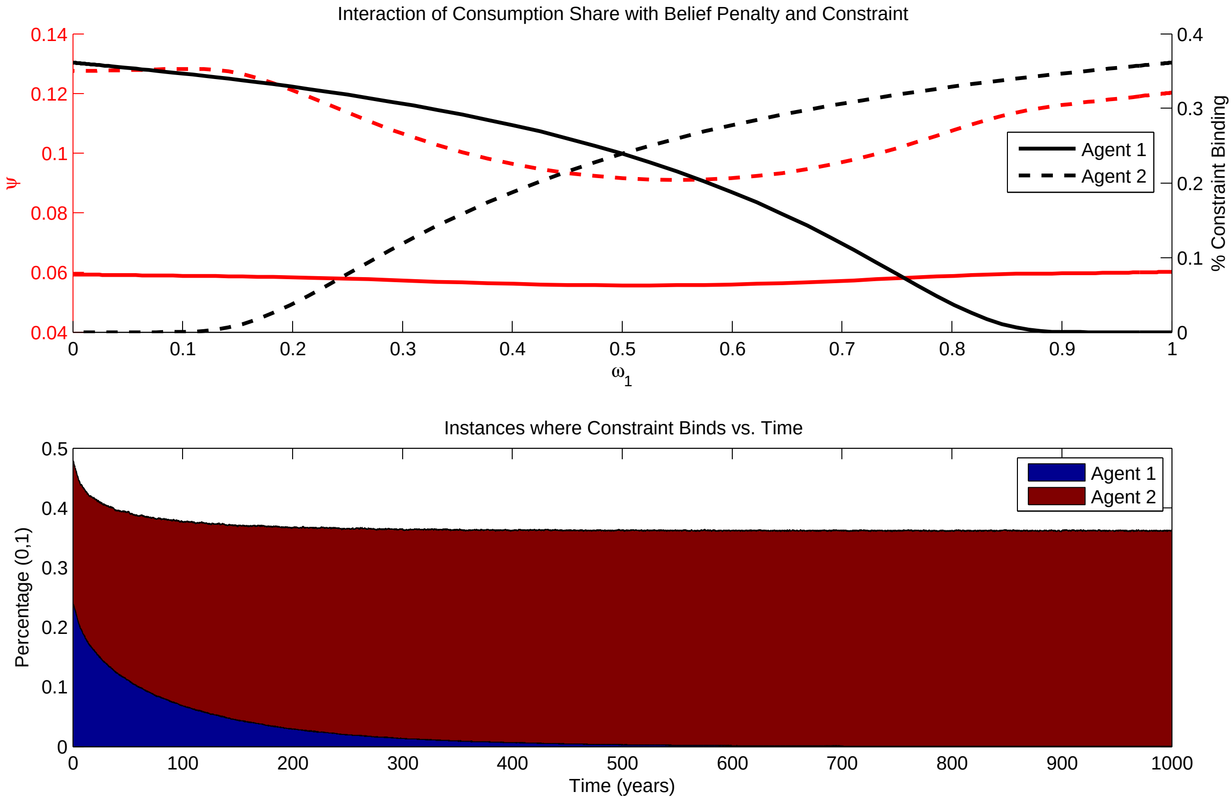

Figure 4: Frequency and effect of constraint binding.
Maximum standard error for ψ over the interval and all parameter combinations is 4.20×10^{-4}.

The bottom plot in Figure 4 illustrates the evolution of the constraint's effects over time under the baseline parameters. As the overconfident agent loses his endowment, he is more likely to be bound by the constraint. The rational agent is correspondingly less likely to be bound, and so the total percentage of paths for which the constraint binds one agent or the other remains roughly unchanged.

When short-sales are restricted, the interaction between ω_i and ψ_i is sufficiently complex that an agent's steady state survival index $\bar{I}_i$ cannot be determined in isolation. Rather, it is only established when an agent's equilibrium ω_i is attained. Hence it is no longer possible to "rank" agents by survival index without solving for the long run equilibrium, reducing the index's usefulness as a "forecasting" tool. However, it remains useful for interpreting observed changes in consumption share.

5.5 Interest Rates and Prices of Risk

When agents are allowed to sell short, $E[\kappa_i(t)] = E\left[\frac{f_i(t)-r(t)}{\sigma_\delta}\right]$. Because all agents agree on $r(t)$ and $E[f_i(t)] = \bar{f}$, $\forall i$, $E[\kappa_i(t)]$ is unaffected by the level of overconfidence. With the introduction of the short-sale constraint, this relationship no longer holds. Accordingly, I focus my analysis on the *ratios* of the constrained to unconstrained $E[\kappa_i(t)]$ and $E[r(t)]$. Figure 5 illustrates the evolution of these ratios. The terminal period is set at 100 years, since the most dramatic changes occur relatively early on, as the highly overconfident agent loses wealth rapidly.

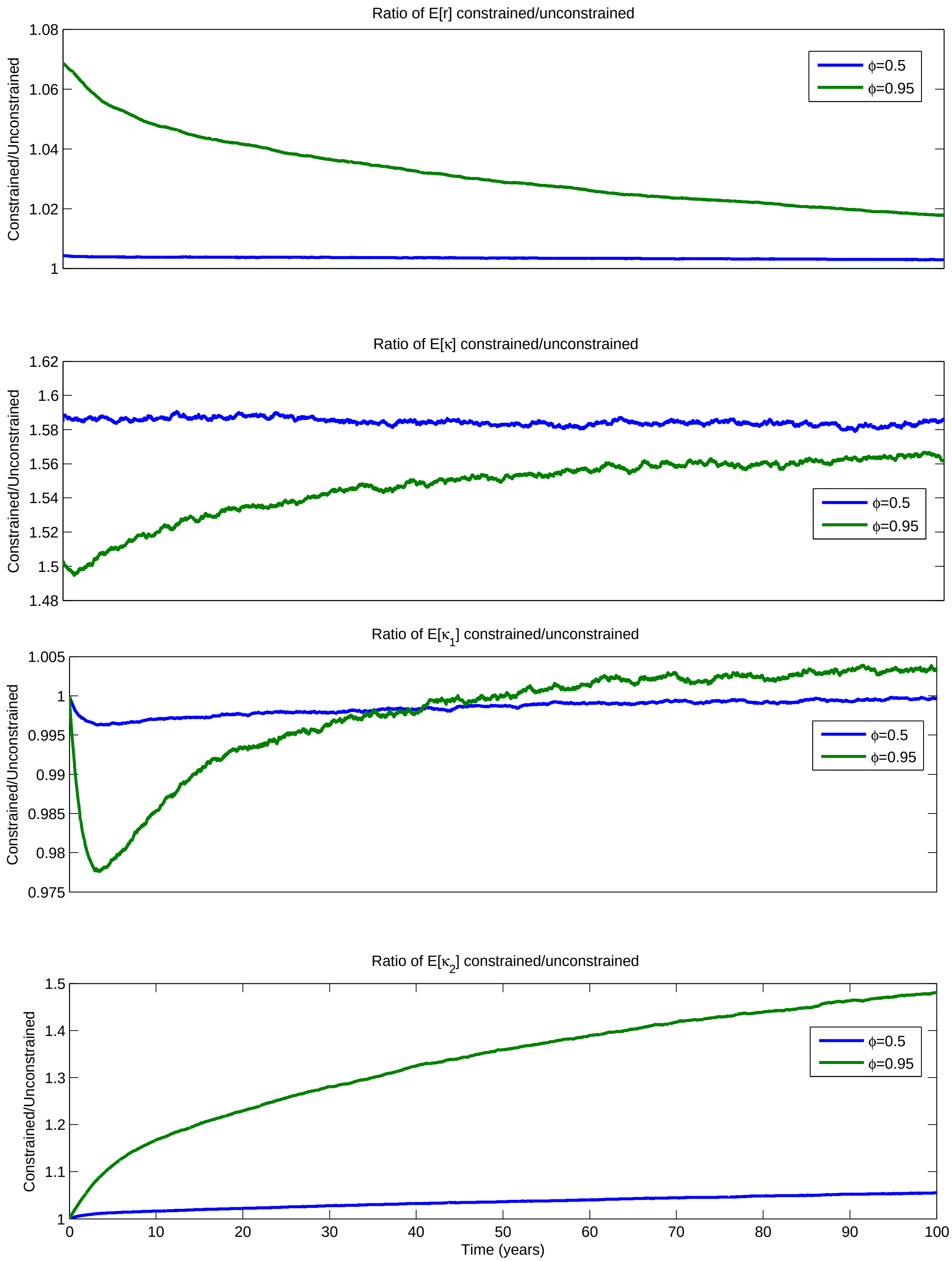

Figure 5: Interaction of constraint with interest rate and price of risk.
Unconstrained expectations are constant at $E[r] = 0.098$, $E[\kappa] = E[\kappa_1] = E[\kappa_2] = 0.13$ for all values of ϕ. Note that the scales along the y-axes of the charts for κ_i vary dramatically. Maximum standard errors over the interval and all parameter combinations are as follows: $E[r] : 4.47 \times 10^{-5}$, $E[\kappa] : 5.91 \times 10^{-4}$, $E[\kappa_1] : 2.89 \times 10^{-4}$, $E[\kappa_2] : 5.52 \times 10^{-4}$.

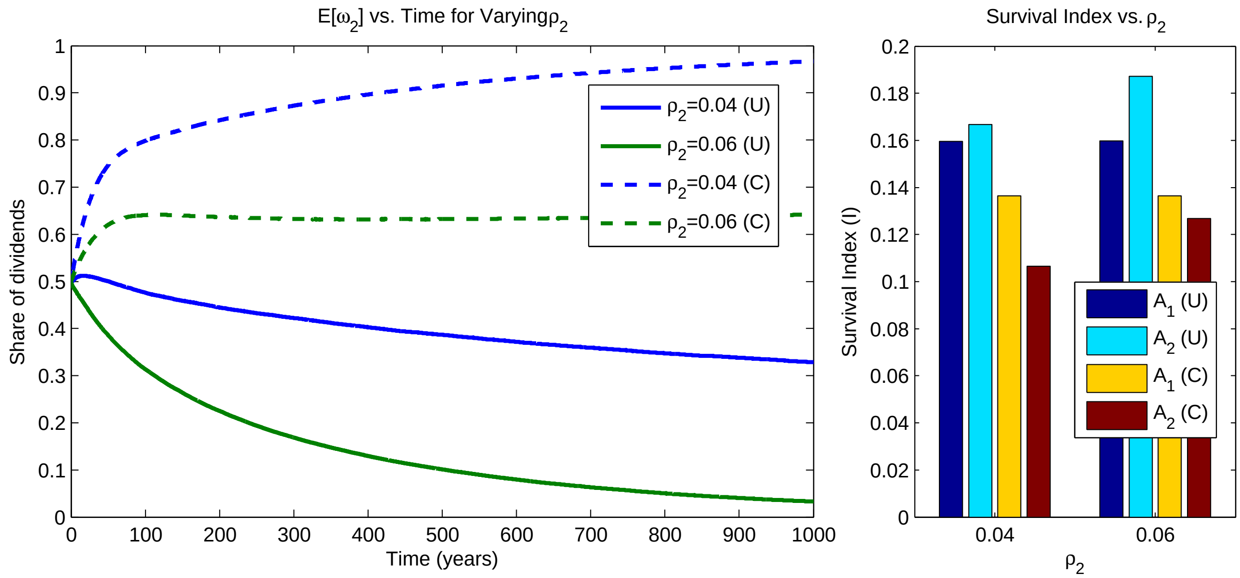

Figure 6: Survival with overconfidence and impatience.
Maximum standard errors over the interval and all parameter combinations are as follows: $E[\omega_2] : 7.10 \times 10^{-4}$, $\psi : 5.52 \times 10^{-4}$

Consistent with discussion in Gallmeyer and Hollifield [2008], imposing the constraint causes the interest rate to increase, significantly for $\phi = 0.95$, but only slightly with $\phi = 0.5$. As agent 1 increases his endowment share, his beliefs have a growing influence upon the equilibrium interest rate, and he is bound less frequently by the constraint. As a result, $E[r(t)]$ gradually declines towards its unconstrained level.

Interpreting the price of risk is somewhat more complicated. As the overconfident agent loses his endowment, his beliefs have reduced impact upon $r(t)$, and $\kappa_2(t)$ is more likely to be bound by the constraint. Effectively, the variance of $\kappa_2(t)$ increases as $\omega_2(t)$ declines, but all realizations of $\kappa_2(t)$ below zero are truncated. This causes $E[\kappa_2]$ to rise. $\kappa_1(t)$ is subject to two competing effects. As $\omega_1(t)$ increases, the constraint is more likely to bind when agent 1 is optimistic (i.e., it binds agent 2). In this case, $\kappa_1(t)$ will decrease as $r(t)$ increases. As the overconfident agent's influence wanes, however, $r(t)$ will change little when agent 2 is unable to sell short. We see for $\phi = 0.95$ that the first effect dominates initally, with $E[\kappa_1(t)]$ decreasing in the first few years. Later the second effect dominates, and $E[\kappa_1(t)]$ increases.

5.6 The Relative Importance of Overconfidence

In a his constant growth economy, Yan [2008] observed that even significant errors in an agent's belief about the growth rate can be dwarfed by relatively small differences in impatience or risk aversion, and that estimates of the latter two parameters are widely dispersed. In his setup, a rational agent who knows the correct constant growth rate of $\tilde{f}$ would be dominated by an irrational agent who perennially assumes growth $1.25\tilde{f}$ but has impatience just 0.01 smaller than the rational agent's. However, stochastic growth arguably increases what error in beliefs can be considered reasonable, since even the rational agent will never cease to make errors.

In Figure 6, the relative importance of errors in belief is assessed by introducing heterogeneous impatience. Agent 1 retains the baseline value $\rho_1 = 0.1$, while agent 2's impatience is decreased. So long as agents are able to sell short, the magnitude of $\psi_1 - \psi_2$ is sufficient that agent 2 has a larger survival index despite his patient nature. When short-sales are disallowed, however, belief penalties are reduced, and the overconfident agent's superior (lower) patience parameter gives him the smaller survival index. Although $\psi_1 < \psi_2$ even under the constraint, agent 2 dominates, since $\bar{I}_1 > \bar{I}_2$. If agents are likely to face obstacles to selling short, then overconfidence is arguably less significant than other factors in determining survival. This supports Yan's suggestion that erroneous beliefs may be of secondary importance to survival, given the wide dispersion of estimates for impatience and risk aversion.

Finally, the path of agent 2's consumption share under the constraint illustrates the connection between ω_2 and ψ_2 that was discussed earlier. The belief penalty takes a minimum around $\omega_2(t) = 0.5$, i.e., at time zero. As his consumption share increases, his penalty also increases, and $I_2(t)$ approaches $I_1(t)$, slowing the growth of agent 2's endowment. The effect is particularly noticeable for $\rho_2 = 0.6$.

6 Conclusions

When short-sales are constrained, even very overconfident investors may survive for over a thousand years. Overconfident agents who are modestly more patient than their rational competitors may even come to dominate the market. This bolsters the argument that overconfident agents can survive long enough in a competitive environment to affect stock prices and volatility, offering possible explanations for observed phenomena in asset prices.

In addition, I have examined the dynamic effects of a short-sale constraint when agents are not statically optimistic or pessimistic, but change their minds about the economy from time to time. Complex interactions between wealth, beliefs, and the constraint are distilled somewhat.

The economy and the overconfident agent offer several opportunities for future work. In particular, there are many kinds of market frictions. My work could be combined with that of Longstaff [2009] on illiquidity, or perhaps transaction costs could be modeled. Analytical solutions for the growth forecast error of the overconfident agent could shed more light on his behavior under various economic parameters. Finally, there are opportunities for empirical work in estimating whether (and how) overconfident agents could explain observed asset pricing anomalies.

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