The Price of Oil Risk

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Abstract We solve a Pareto risk-sharing problem for two agents with heterogeneous recursive utility over two goods: oil, and a general consumption good. Using the optimal consumption allocation, we derive a pricing kernel and the price of oil and related futures contracts. This gives us insight into the dynamics of prices and risk premia. We compute portfolios that implement the optimal consumption policies, and demonstrate that large and variable open interest is a property of optimal risk-sharing. A numerical example of our model shows that rising open interest and falling oil risk premium are an outcome of the dynamic properties of the optimal risk sharing solution.

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1 Introduction

The spot price of crude oil, and commodities in general, experienced a dramatic price increase in the summer of 2008. For oil, the spot price peaked in early July 2008 at $145.31 per barrel (see Figure 1). In real-terms, this price spike exceeded both of the OPEC price shocks of 1970's and has lasted much longer than the price spike at the time of the Iraq invasion of Kuwait in the summer of 1990. The run-up to the July 2008 price of oil begins around 2004. Buyuksahin, Haigh, Harris, Overdahl, and Robe (2011) and Hamilton and Wu (2014) identify a structural change in the behavior of oil prices around 2004. This 2004 to 2008 time period also coincides with a large increase in trading activity in commodities by hedge funds and other financial firms, as well as a growing popularity of commodity index funds (see, e.g., CFTC (2008)). In fact, there is much in the popular press that laid the blame for higher commodity prices, food in particular, on the "financialization" of commodities.[1] Others suggest that since these new traders in futures do not end up consuming any of the spot commodity, the trading can have little (if any) effect on spot prices.[2] Resolving this debate requires modeling the equilibrium relationship between spot and futures prices. It also requires a clearer understanding of hedging and speculation. To achieve this, we look directly at the risk-sharing Pareto problem in an economy with heterogenous agents and multiple goods, and solve for equilibrium risk premia.

Our intuition about the use and pricing of commodity futures contracts is often expressed with hedgers and speculators. This dates back to Keynes (1936) and his discussion of "normal backwardation" in commodity markets. The term backwardation is used in two closely related contexts. Often it is used to refer to a negatively sloped futures curve where, e.g., the one-year futures price is below the current spot price. Keynes' use of the term

[1] See for example "The food bubble:How Wall Street starved millions and got away with it" by Frederick Kaufman, Harpers July 2010 http://harpers.org/archive/2010/07/0083022

[2] The clearest argument along this lines is by James Hamilton http://www.econbrowser.com/archives/2011/08/fundamentals_sp.html. See also Hamilton (2009) and Wright (2011)

"normal backwardation" (or "natural") refers to the situation where the current one-year futures price is below the expected spot price in one-year. This you will recognize as a risk premium for bearing the commodity price risk. This relation is "normal" if there are more hedgers than there are speculators. Speculators earn the risk premium, and hedgers benefit from off-loading the commodity price risk. But there is no reason to assume that hedgers are only on one side of the market. Both oil producers (Exxon) and oil consumers (Southwest Air) might hedge oil. In oil markets in the 2004 to 2008 period, there was a large increase on the long-side by speculators suggesting the net "commercial" or hedging demand was on the short side. This is documented in Buyuksahin, Haigh, Harris, Overdahl, and Robe (2011), who use proprietary data from the CFTC that identifies individual traders. However, if we are interested in risk premia in equilibrium we need to look past the corporate form of who is trading. We own a portfolio that includes Exxon, Southwest Air, and a commodity hedge fund; we consume goods that, to varying degrees, depend on oil. Are we hedgers or speculators?

Equilibrium risk premia depend on preferences, endowments, technologies, and financial markets. In this model we focus on complete and frictionless financial markets. We leave aside production, and consider an endowment economy with two goods. One good we calibrate to capture the salient properties of oil; the other we think of as a composite good akin to consumption in the macro data. We consider two agents with heterogenous preferences over the two goods, as well as with different time and risk aggregators (using Epstein and Zin (1989) preferences). Preference heterogeneity is a natural explanation for the portfolio heterogeneity we see in commodity markets. Here we start with complete and frictionless markets, focus on "perfect" risk sharing, and solve for the Pareto optimal consumption allocations. From this solution, we can infer the "representative agent" marginal rates of substitution, and calculate asset prices and the implied risk premiums.

To see how heterogeneity plays a role, consider an agent who weighs two severe risks: a recession caused by an oil crisis, and a recession with another root cause – say a financial crisis. Both scenarios are high marginal-rate-of-substitution states. However, in the oil-lead recession, the price of oil is high. In the financial-crisis-lead recession the oil price is low. Which is worse? The answer, which depends on preferences, determines the agent's attitude towards a short-dated crude oil futures contract, and ultimately determines the risk premium for a long position in oil. In our numerical example, in Section 4, we have two agents who rank these scenarios differently. The risk premium on oil depends on the relative wealth of the two agents. Each agent holds a Pareto-optimal portfolio, but realized returns may increase the wealth of one agent versus the other. So the longer horizon risk premiums and their dynamics over time depend on the evolution of the wealth shares. Because they occur endogenously, changes in the wealth distribution also provide an alternative (or complementary) explanation for persistent changes in oil markets that does not rely upon exogenously imposed structural breaks, such as permanent alterations to the consumption growth process, or changing access to financial markets.

Models of commodity futures typically study either active trade and risk sharing, or cross-market equilibrium price dynamics, but not both. For example, work in the spirit of the normal backwardation theory of Keynes and Hicks, such as Hirshleifer (1988) or more recently Baker (2015), allows for hedging using commodity futures, but assumes that revenue streams are generally non-marketable: agents are either commodity producers or commodity consumers, by virtue of their endowments. By contrast other studies situate commodity futures in complete financial markets, but consider the dual problems of either a representative producer (as in Kogan, Livdan, and Yaron (2009)) or a representative household (as in Ready (2014)), and do not explicitly model trade in futures. Hitzemann (2015) studies a general equilibrium model with a representative household and a firm that produces and stores oil, but also without active trade in futures. An exception is Hirshleifer (1990), in which households receive heterogeneous endowment streams of two goods, and optimally hedge by trading futures. The elegant theoretical setting, without preference heterogeneity or intertemporal consumption, is sufficient to demonstrate that futures risk premia are not generally a function of hedging pressure. However these simplifying assumptions render study of joint spot price and futures open interest dynamics impossible, and the lack of preference heterogeneity limits the impact of equilibrium wealth dynamics on commodity prices and risk premia. To overcome this limitation we build on many papers that look at risk sharing and models with heterogenous agents. We are most closely building on Backus, Routledge, and Zin (2009) and (2008), with a model structure similar to Colacito and Croce (2014), but with fewer restrictions on preference parameters. Foundational work in risk sharing with recursive preferences includes Lucas and Stokey (1984), Kan (1995), and Anderson (2005).

The bulk of our paper explores a numerical example of our model. The example demonstrates how dynamic risk sharing between agents with different preferences generates wide variation in prices, risk premia, and open interest over time. This frictionless Pareto benchmark sheds some light on why empirical studies have reached somewhat nuanced conclusions regarding the connection between futures open interest, spot prices, and financial asset returns. For example Hong and Yogo (2012) find that futures open interest is a good predictor of commodity and bond returns. A series of papers, including Stoll and Whaley (2010), Buyuksahin, Haigh, Harris, Overdahl, and Robe (2011), Hamilton and Wu (2014), and Singleton (2014), discuss the "financialization" of commodity futures markets documented since around 2004, illustrated by an increase in the number of contracts traded by commodity index funds specifically. Stoll and Whaley (2010) find that lagged futures returns drive out the predictive power of commodity index flows, whereas Singleton (2014) finds the flows have predictive power after controlling for lagged returns and open interest generally. As summarized in Irwin and Sanders (2011), evidence on causal linkages between positions and futures return moments is mixed. Kilian and Murphy (2014) argue that global demand shocks rather than speculative futures trade explained the 2003-2008 oil price surge. Financial trade and consumer demand are impossible to disentangle in our complete markets equilibrium setting, because the wealth dynamics that ultimately drive demand are dependent upon financial market trade. Yet the relationship between open interest and consumer demand is relatively flat for a range of the wealth distribution.

We begin by summarizing several facts regarding crude oil futures returns and US treasury bond returns. We then present our theoretical model, and solve for several useful results. A general analytical model solution is not possible; however the model can be solved numerically in terms of a small number of state variables. We discuss a numerical example in terms of two main state variables - the economic growth state and a proxy for the wealth distribution - and also compare the model to data based on observable characteristics such as the slope of the futures curve and open interest.

2 Facts

We are interested in time variation in expected excess returns to a long position in crude oil futures. Since a futures contract is a zero-wealth position, we define a fully collateralized return as follows. $F_{t,n}$ is the futures price at date t for delivery at date $t+n$, with the usual boundary condition that the $n = 0$ contract is the spot price of oil: $F_{t,0} = P_t$. The fully collateralized return involves purchasing $F_{t,n}$ of a one-month bond and entering into the $t + n$ futures contract with agreed price $F_{t,n}$ at date t. Cash-flows at date $t + 1$ come from the one-period risk free rate, $r_{f,t+1}$ and the change in futures prices $F_{t+1,n-1} - F_{t,n}$. So,

$$r^n_{oil,t+1} \quad = \quad \log\left(\frac{F_{t+1,n-1} - F_{t,n} + (F_{t,n}(\exp r_{f,t+1}))}{F_{t,n}}\right). \tag{1}$$

Excess returns are approximately equal to the log-change in futures prices,

$$r^n_{oil,t+1} - r_{f,t+1} \quad \approx \quad \log F_{t+1,n-1} - \log F_{t,n}. \tag{2}$$

We measure the risk-premium on each contract as the average log-change.

We use one-month to the 60-month futures contracts for WTI light-sweet crude oil traded at NYMEX from Jan. 1990 through April 2016.[3] To generate monthly data, we use the price on the last trading day of each month. The liquidity and trading volume is higher in near-term contracts. However, oil has a reasonably liquid market even at the longer horizons, such as out to the 60-month contract. For long term contracts not listed in every month, we use the return on the nearest available contract exceeding the stated horizon.

Many models of crude oil storage or production dynamics point to the slope of the futures curve as an important (endogenous) variable, e.g., Carlson, Khokher, and Titman (2007), Casassus, Collin-Dufresne, and Routledge (2007), Kogan, Livdan, and Yaron (2009), and Routledge, Seppi, and Spatt (2000). To investigate in the data, we define the slope as the log-difference of the 18-month and nearest futures contracts,

$$slope_t = \log F_{t,18} - \log F_{t,1}, \tag{3}$$

and document properties of excess returns conditional on the previous month's slope.[4] Figure 2 highlights that, indeed, this state variable is important to the dynamics of the risk premium associated with a long position in oil. Average excess returns are around 1% per month for all contracts when the slope is negative, but around -0.5% per month or less when the slope is positive. The standard deviations of excess returns do not vary dramatically with the slope, but are around 1% per month lower for most contracts when the slope is negative.

While Figure 2 suggests a connection between the slope and the risk premium, our estimates of conditional average excess returns come with a good deal of noise. To assess statistical significance, we estimate coefficients of the regression

$$r^n_{oil,t+1} - r^f_{t+1} = \beta_0 + \beta_1 slope_t + \epsilon_{t+1}, \tag{4}$$

[3]Data was aggregated by Quandl. All the contract details are at http://www.cmegroup.com/trading/energy/crude-oil/light-sweet-crude_contract_specifications.html

[4]Results are similar if using, for example, the 12-month contract instead of the 18-month. The idea here is similar to the forecasting regressions in Fama and French (1987).

for several contracts, and report results in Table 1.[5] Estimated coefficients on the slope are negative for all contracts, and are significant at the 5% level or better for contracts of 18-months or more.[6]

In keeping with studies of commodity financialization such as Hamilton and Wu (2014) and Buyuksahin, Haigh, Harris, Overdahl, and Robe (2011), we also split our data into 1990-2003 and 2004-2016 subsamples, to see if relationships differ between periods. Figure 2 shows in the middle row that the post-2004 period has been one of higher volatility for most contract returns, but average excess returns are about the same for both periods. A reduction in average excess returns on the nearest contracts from 2004 is the most noticeable change. The bottom row of the figure shows that the slope is negatively related to average excess returns in each period, similar to results for the full sample, and the sign of the slope is not associated with large changes in volatility. Repeating our regressions for each time period (omitted) yields similar results for the slope coefficients to those in Table 1. Financialization does not appear to have unhinged the relationship between the slope and the risk premium. However the slope is less frequently negative from 2004, about 41% of the time, versus 69% of the time before 2004, and about 56% for the full sample. Recently, backwardation has not been the norm.

The fact that oil seems to command a risk premium suggests its price is correlated with the pricing kernel, and evidence from the slope suggests time variation the sign of that correlation. Of course, oil is an important commodity directly tied to economic activity. Hamilton (2008) documents that nine out of ten of the U.S. recessions since World War II were preceded by a spike up in oil prices, with the "oil-shocks" of the 1970's as the most dramatic examples. Even the most recent recession, December 2007 to June 2009, coincides with dramatic spike in oil prices. The peak oil price was during the summer of 2008 – right

[5] This regression is similar to the familiar Fama-French regression that predicts the future spot price change with the slope. In that regression, the fact the slope does not one-for-one predict future price changes is evidence of a risk premium that varies with the slope.

[6] Results using Brent crude oil futures are similar.

before the collapse of Lehman Brothers. However, by December of 2007, the WTI spot price was a very high \$91.73 per barrel.

If the crude oil futures slope indirectly reveals information about the pricing kernel, then perhaps the slope is informative of bond risk premia also? To investigate, we estimate the regression

$$r^n_{bond,t+1} - r^f_{t+1} = \beta_0 + \beta_1 slope_t + \beta_2 CP_t + \epsilon_{t+1}, \tag{5}$$

where the left side is the excess return on a treasury bond portfolio with maturity n. We add the factor CP_t from Cochrane and Piazzesi (2005) to the right side, and consider annual log holding period returns formed from monthly CRSP Treasury bond index data from Jan. 1990 through Dec. 2003.[7] Results in Table 2 show that a negative crude oil slope predicts positive excess bond returns, with significance at the 5% level or better, depending on bond maturity. As demonstrated elsewhere, the CP_t factor also predicts excess bond returns during this period. When combined with the slope both factors remain significant, achieving R^2 of around 0.5 for longer maturity bonds. In unreported additional regressions we include summary macroeconomic factors from Ludvigson and Ng (2009), also shown to predict excess Treasury bond returns, and find that the slope coefficient remains significant. Results are sensitive to the holding period however, and are not as strong for, e.g., monthly holding periods. We also included bond factors in the predictive regressions for crude oil futures returns, and found that the bond factors do not significantly predict excess futures returns. Finally, with the slope alone we extend our sample period through April 2016. Table 3 shows that coefficients for most maturities remain significant at the 5% level or better, although results for 1 and 2 year bonds especially are weaker once the 2008 financial crisis is included.

Having established a relationship between the crude oil futures slope and risk premia in futures and Treasuries, what remains is a direct connection to financial trade and risk-

[7]Cochrane and Piazzesi (2005) use Fama-Bliss bond indices, whereas we use similar CRSP Treasury bond index data, which extends to synthetic bonds with longer maturity.

sharing. Important work along these lines is by Hong and Yogo (2012), who show that commodity market open interest predicts excess returns on commodity futures and bonds, and to a lesser degree excess stock and currency returns. They construct a composite measure of growth in open interest across several commodity sectors, whereas we focus here on oil. To illustrate the case for oil specifically, Figure 12 plots open interest and the slope from Jan. 1990 through April 2016. We define OI_t as the number of outstanding futures contracts after removing a linear trend in logs, using data from the CFTC. Open interest and the slope are positively correlated, although OI_t appears to lag $slope_t$, particularly following sharp drops in the slope such as after the 2008 financial crisis.

Table 4 estimates

$$slope_t = \beta_0 + \beta_1 OI_t + \beta_2 CI_t + \epsilon_t, \tag{6}$$

adding CI_t, the detrended percentage difference in commercial long versus short futures contracts. This is similar to the composite commercial imbalance measure of Hong and Yogo (2012). Each measure explains significant variation in the slope, with a combined R^2 of around 20%. Although the slope does appear related to trade, we do not find OI_t or CI_t to be strong direct predictors of excess returns on futures or bonds. This supports the approach of Hong and Yogo (2012) to form smoothed composite measures of open interest with less noise. However our model results will also suggest open interest has a more fragile relationship to the risk premium than does the slope.

3 Model

We model an exchange or endowment economy as in (Lucas 1978). We specify a stochastic process for the endowment growth. Specifically, we will have one good x_t we think of as the "numeraire" or composite commodity good. Our second good, which we calibrate to be oil, we denote y_t. We use the short-hand notation subscript-t to indicate conditional on the history to date t. Similarly, we use E_t and μ_t to indicate expectations and certainty equivalents conditional on information to date-t. Heterogeneity in our setup will be entirely driven by preference parameters. Beliefs across all agents are common.

We are interested in the Pareto optimal allocation or "perfect" risk sharing solution. So with complete and frictionless markets, we focus on the social planner's Pareto problem. This means, for now, we need not specify the initial ownership of the endowment; we treat x_t and y_t as resource constraints. The preferences, which we allow to differ across our two agents, are recursive as in Epstein and Zin (1989) and Kreps and Porteus (1978). They are characterized by three "aggregators" (see Backus, Routledge, and Zin (2005)). First, a goods aggregator determines the tradeoff between our two goods. This is, of course, a simplification since oil is not directly consumed. But the heterogeneity across our two agents will capture that some of us are more reliant on the consumption of energy-intensive products. The other two aggregators are the usual time aggregator and risk aggregator that determine intertemporal substitution and risk aversion. Finally, the familiar time-additive expected utility preferences are a special case of this setup.

3.1 Single Agent, Two Goods

To get started, consider a single-agent economy with two goods. In this setting the representative agent consumes the endowment x_t and y_t each period. We model utility from consumption of the "aggregated good" with a Cobb-Douglas aggregator,

$$A_t = A(x_t, y_t) = x_t^{1-\gamma} y_t^{\gamma},$$

with $\gamma \in [0, 1]$. The agent has Epstein-Zin recursive preferences over consumption of the aggregated good,

$$\begin{aligned} W_t = W(x_t, y_t, W_{t+1}) &= \left[(1-\beta)A(x_t, y_t)^{\rho} + \beta \mu_t(W_{t+1})^{\rho}\right]^{1/\rho}, \\ \mu_t(W_{t+1}) &= E_t\left[W_{t+1}^{\alpha}\right]^{1/\alpha}. \end{aligned}$$

Endowment growth follows a finite-state Markov process. Denote the state s_t, with the probability of transitioning to next period state s_{t+1} given by $\pi(s_t, s_{t+1})$, for $s_t, s_{t+1} \in S$ with $|S|$ finite. Growth in the numeraire good is $f_{t+1} = f(s_{t+1}) = x_{t+1}/x_t$, and similarly for the oil-good $g_{t+1} = g(s_{t+1}) = y_{t+1}/y_t$.

With a little algebra, we can write the pricing kernel as

$$m_{t+1} = \frac{\partial W_t/\partial x_{t+1}}{\partial W_t/\partial x_t}(\pi_{t+1})^{-1} = \beta \left(\frac{x_{t+1}}{x_t}\right)^{-1} \left(\frac{A_{t+1}}{A_t}\right)^{\rho} \left(\frac{W_{t+1}}{\mu_t(W_{t+1})}\right)^{\alpha-\rho}. \tag{7}$$

Note that this is denominated in terms of the numeraire good (x). We can use m_{t+1} to compute the price at t of arbitrary numeraire-denominated contingent claims that pay off at $t+1$. Claims to oil good y at t are converted to contemporaneous numeraire values using the spot price of oil,

$$P_t = \frac{\partial W_t/\partial y_t}{\partial W_t/\partial x_t} = \frac{\gamma x_t}{(1-\gamma)y_t}. \tag{8}$$

The pricing kernel and spot price can be used in combination to price arbitrary contingent claims to either good.

The homogeneity of the Cobb-Douglas aggregator along with the standard homogeneity of the time and risk aggregators allow us to rescale things so utility is stationary, similar to Hansen, Heaton, and Li (2008), defining

$$\hat{W}_t = \frac{W_t}{A(x_t, y_t)} = \left[(1-\beta) + \beta\mu_t\left(\hat{W}_{t+1}A(f_{t+1}, g_{t+1})\right)^{\rho}\right]^{1/\rho}.$$

This uses the Cobb-Douglas property that $\frac{A(x_{t+1},y_{t+1})}{A(x_t,y_t)} = A(f_{t+1}, g_{t+1})$. Written in this form, $\hat{W}_t$ is stationary and is a function only of the current state s_t. Similarly, substituting $\hat{W}_t$ into the pricing kernel, we have

$$m_{t+1} = \beta\,(f_{t+1})^{-1}\,(A(f_{t+1}, g_{t+1}))^{\rho} \left(\frac{A(f_{t+1}, g_{t+1})\hat{W}_{t+1}}{\mu_t(A(f_{t+1}, g_{t+1})\hat{W}_{t+1})}\right)^{\alpha-\rho}. \tag{9}$$

The pricing kernel depends on the current state s_t, via the conditional expectation in the risk aggregator, and $t+1$ growth state s_{t+1}.

The price of oil depends on the relative levels of the two goods. However, changes in the oil price will depend only on the relative growth rates:

$$\frac{P_{t+1}}{P_t} = \frac{f_{t+1}}{g_{t+1}}.$$

This implies the change in price depends only on the growth state s_{t+1}.

The advantages and limitations of a single agent single good representative agent model are quite well known. For example, with a thoughtfully chosen consumption growth process one can capture many salient feature of equity and bond markets (Bansal and Yaron (2004)). Alternatively, one can look at more sophisticated aggregators or risk to match return moments (Routledge and Zin (2010)). One could take a similar approach to extend to a two-good case to look at oil prices and risk premia; see, e.g., Ready (2010). It would require some work in our specific setup, since oil price dynamics would simply depend on the growth state s_t in combination with constant preference parameter γ.

Instead we introduce a second, but similar, agent. The dynamics of the risk sharing problem we discuss next will provide us a second state variable, besides s_{t+1}, to generate realistic time variation in the oil risk premium. This also lets us look at the portfolios and trades the two agents choose to make. Lastly, note that the single agent case in this section corresponds to the boundary cases in the two-agent economy, where one agent receives zero Pareto weight (or has no wealth).

3.2 Two Agents, Two Goods

Consider a model with two agents, leaving the two-good endowment process and recursive preference structure unchanged. We allow the two agents to have differing parameters for their goods, risk, and time aggregators. Denote the two agents "1" and "2"; these subscripts will denote the preference heterogenity and the endogenous goods allocations. The risk sharing or Pareto problem for the two agents is to allocate consumption of the two goods across the two agents, such that $c^x_{1,t} + c^x_{2,t} = x_t$ and $c^y_{1,t} + c^y_{2,t} = y_t$. Agent one derives utility from consumption of the aggregated good $A_1(c^x_{1,t}, c^y_{1,t}) = (c^x_{1,t})^{1-\gamma_1}(c^y_{1,t})^{\gamma_1}$. The utility from the stochastic stream of this aggregated good has the same recursive form as above,

$$\begin{aligned} W_t = W(c^x_{1,t}, c^y_{1,t}, W_{t+1}) &= \left[(1-\beta)A_1(c^x_{1,t}, c^y_{1,t})^{\rho_1} + \beta\mu_{1,t}(W_{t+1})^{\rho_1}\right]^{1/\rho_1}, \\ \mu_{1,t}(W_{t+1}) &= E_t\left[W^{\alpha_1}_{t+1}\right]^{1/\alpha_1}. \end{aligned} \tag{10}$$

Agent two has similar preference structure with $A_2(c^x_{2,t}, c^y_{2,t}) = (c^x_{2,t})^{1-\gamma_2}(c^y_{2,t})^{\gamma_2}$ and recursive preferences

$$\begin{aligned} V_t = V(c^x_{2,t}, c^y_{2,t}, V_{t+1}) &= \left[(1-\beta)A_2(c^x_{2,t}, c^y_{2,t})^{\rho_2} + \beta\mu_{2,t}(V_{t+1})^{\rho_2}\right]^{1/\rho_2}, \\ \mu_{2,t}(V_{t+1}) &= E_t\left[V^{\alpha_2}_{t+1}\right]^{1/\alpha_2}. \end{aligned} \tag{11}$$

The idea is that the two agents can differ about the relative importance of the oil good, risk aversion over the "utility lotteries", or the intertemporal smoothing. Recall that with recursive preferences all of these parameters will determine the evaluation of a consumption bundle. "Oil risk" does not just depend on the γ parameter since it involves an intertemporal, risky consumption lottery. We give the two agents common rate of time preference β.[8]

The two-agent Pareto problem is a sequence of consumption allocations for each agent $\{c^x_{1,t}, c^y_{1,t}, c^x_{2,t}, c^y_{2,t}\}$ that maximizes the weighted average of date-0 utilities subject to the aggregate resource constraint which binds at each date and state:

$$\begin{aligned} \max_{\{c^x_{1,t}, c^y_{1,t}, c^x_{2,t}, c^y_{2,t}\}} \lambda W_0 &\quad + \quad (1-\lambda)V_0 \\ \text{s.t.}\ c^x_{1,t} + c^x_{2,t} &\quad = \quad x_t \quad \text{and} \\ c^y_{1,t} + c^y_{2,t} &\quad = \quad y_t \quad \text{for all } s^t, \end{aligned}$$

where λ determines the relative importance (or date-0 wealth) of the two agents. Even though each agent has recursive utility, the objective function of the social planner is not

[8] Differing β's are easy to accommodate but lead to uninteresting models since the agent with the larger β quickly dominates the optimal allocation. See, e.g., Yan (2008).

recursive, except in the case of time-additive expected utility. We can rewrite this as a recursive optimization problem, following Lucas and Stokey (1984) and Kan (1995):

$$\begin{aligned} J(x_t, y_t, V_t) = \max_{c^x_{1,t}, c^y_{1,t}, V_{t+1}} & \left[(1-\beta)A_1(c^x_{1,t}, c^y_{1,t})^{\rho_1} + \beta\mu_{1,t}(J(x_{t+1}, y_{t+1}, V_{t+1}))^{\rho_1}\right]^{1/\rho_1} \\ \text{s.t. } & V(x_t - c^x_{1,t}, y_t - c^y_{1,t}, V_{t+1}) \geq V_t. \end{aligned} \tag{12}$$

The optimal policy involves choosing agent one's date t consumption, $c^x_{1,t}$, $c^y_{1,t}$ and the resource constraint pins down agent two's date t bundle. In addition, at date t we solve for date $t+1$ "promised utility" for agent two. This promised utility is a vector, since we choose one for each possible growth state s_{t+1}. Making good on these promises at date $t+1$ means that V_{t+1} is an endogenous state variable we need to track. That is, optimal consumption at date t depends on the exogenous growth state s_t and the previously promised utility V_t. Finally, note that the solution to this problem is "perfect" or optimal risk sharing. Since we consider complete and frictionless markets, there is no need to specify the individual endowment process.

Preferences are monotonic, so the utility-promise constraint will bind. Therefore with optimized values, we have $W_t = W(c^x_{1,t}, c^y_{1,t}, W_{t+1}) = J(x_t, y_t, V_t)$ and $V_t = V(x_t - c^x_{1,t}, y_t - c^y_{1,t}, V_{t+1})$. Optimality conditions imply that the marginal utilities of agent one and agent two are aligned across goods and intertemporally:

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c^x_{1,t+1}}{c^x_{1,t}}\right)^{-1} \left(\frac{A_{1,t+1}}{A_{1,t}}\right)^{\rho_1} \left(\frac{W_{t+1}}{\mu_{1,t}(W_{t+1})}\right)^{\alpha_1-\rho_1} \\ &= \beta \left(\frac{c^x_{2,t+1}}{c^x_{2,t}}\right)^{-1} \left(\frac{A_{2,t+1}}{A_{2,t}}\right)^{\rho_2} \left(\frac{V_{t+1}}{\mu_{2,t}(V_{t+1})}\right)^{\alpha_2-\rho_2}. \end{aligned} \tag{13}$$

Recall that beliefs are common across the two agents so probabilities drop out. We can use this marginal-utility process as a pricing kernel. Optimality implies agents agree on the price of any asset.

Similarly, the first-order conditions imply agreement about the intra-temporal trade of the numeraire good for the oil good. Hence the spot price of oil:

$$P_t = \frac{\gamma_1 c^x_{1,t}}{(1-\gamma_1)c^y_{1,t}} = \frac{\gamma_2 c^x_{2,t}}{(1-\gamma_2)c^y_{2,t}}. \tag{14}$$

As in the single agent model, homogeneity allows for convenient rescaling. The analogous scaling in the two-agent setting is

$$\hat{c}^x_{1,t} = \frac{c^x_{1,t}}{x_t} \quad , \quad \hat{c}^y_{2,t} = \frac{c^x_{2,t}}{x_t} = 1 - \hat{c}^x_{1,t},$$
$$\hat{c}^y_{1,t} = \frac{c^y_{1,t}}{y_t} \quad , \quad \hat{c}^y_{2,t} = \frac{c^y_{2,t}}{y_t} = 1 - \hat{c}^y_{1,t}.$$

The $\hat{c}$'s are consumption shares of the two goods. We also scale utility values using their respective goods aggregators,

$$\hat{W}_t = \frac{W_t}{A_1(x_t, y_t)} \quad , \quad \hat{V}_t = \frac{V_t}{A_2(x_t, y_t)}.$$

Notice we scale the utilities by the total available goods, not just the agent's share. This has the advantage of being robust if one agent happens to (optimally) get a declining share of consumption over time.[9] Plugging these into the equation (13), and we can state the pricing kernel as

$$m_{t+1} = \beta \left(\frac{f_{t+1}\hat{c}^x_{1,t+1}}{\hat{c}^x_{1,t}} \right)^{-1} \left(\frac{A_1(f_{t+1}\hat{c}^x_{1,t+1}, g_{t+1}\hat{c}^y_{1,t+1})}{A_1(\hat{c}^x_{1,t}, \hat{c}^y_{1,t})} \right)^{\rho_1} \left(\frac{A_1(f_{t+1}, g_{t+1})\hat{W}_{t+1}}{\mu_1(A_1(f_{t+1}, g_{t+1})\hat{W}_{t+1})} \right)^{\alpha_1 - \rho_1}, \tag{15}$$

or equivalently from the perspective of agent two. In the one-agent case, the pricing kernel depends on the current growth state s_t and the future growth state s_{t+1}. In the two agent case, the pricing kernel depends on both the growth state and the scaled utility of agent two currently $(s_t, \hat{V}_t)$ and in the future $(s_{t+1}, \hat{V}_{t+1})$.

3.3 Financial Prices

We can now use the pricing kernel to price assets and calculate their returns. To start, we can look at the values of each agent's consumption streams, which measure individual wealth. Agent one's claim to numeraire consumption good x has value at date t

$$C^x_{1,t} = E_t \left[\sum_{\tau=t}^{\infty} m_\tau c^x_{1,\tau} \right]. \tag{16}$$

[9] In particular, we use these normalizations to make the optimization problem in Equation (12) independent of current levels of x_t and y_t.

This is the *cum* dividend value, including current consumption.

We conjecture that

$$C^x_{1,t} = \frac{c^x_{1,t} W_t^{\rho_1}}{(1-\beta) A_{1,t}^{\rho_1}}, \tag{17}$$

which is easily verified using the pricing kernel definition in equation (13). Substituting our normalizations,

$$C^x_{1,t} = \left(\frac{\hat{c}^x_{1,t} \hat{W}_t^{\rho_1}}{(1-\beta) A_1(\hat{c}^x_{1,t}, \hat{c}^y_{1,t})^{\rho_1}} \right) x_t. \tag{18}$$

Note that the price of the claim to numeraire consumption is conditionally independent of the level of aggregate oil consumption; that is, the ratio $C^x_{1,t}/x_t$ depends only on our state variables s_t and $\hat{V}_t$. To price the claim to the oil consumption good, we use the oil price to convert to units of numeraire good:

$$C^y_{1,t} = E_t \left[\sum_{\tau=t}^{\infty} m_\tau P_\tau c^y_{1,\tau} \right] = \frac{\gamma_1}{1-\gamma_1} C^x_{1,t}.$$

Again, aggregate oil consumption does not play a role, and the ratio $C^y_{1,t}/x_t$ depends only on the state variables s_t and V_t. Lastly, summing the value of the numeraire and oil claim we calculate the total wealth of agent one:

$$C_{1,t} = C^x_{1,t} + C^y_{1,t} = \frac{1}{1-\gamma_1} C^x_{1,t}. \tag{19}$$

By equivalent logic, the values of agent two's consumption claims are

$$\begin{aligned} C^x_{2,t} &= \left(\frac{\hat{c}^x_{2,t} \hat{V}_t^{\rho_2}}{(1-\beta) A_2(\hat{c}^x_{2,t}, \hat{c}^y_{2,t})^{\rho_2}} \right) x_t. \\ C^y_{2,t} &= \frac{\gamma_2 C^x_{2,t}}{1-\gamma_2} \\ C_{2,t} &= \frac{C^x_{2,t}}{1-\gamma_2} \end{aligned}$$

With the wealth of each agent, we can define aggregate wealth in the numeraire sector,

$$C^x_t = C^x_{1,t} + C^x_{2,t}, \tag{20}$$

in the oil sector,

$$C_t^y = \frac{\gamma_1 C_{1,t}^x}{1-\gamma_1} + \frac{\gamma_2 C_{2,t}^x}{1-\gamma_2}, \tag{21}$$

and the overall wealth in the economy,

$$C_t = \frac{C_{1,t}^x}{1-\gamma_1} + \frac{C_{2,t}^x}{1-\gamma_2}. \tag{22}$$

Bond prices and the risk free rate all follow from the pricing kernel in the usual way. Define the price of a zero-coupon bond recursively as

$$B_{t,n} = E_t[m_{t+1} B_{t+1,n-1}], \tag{23}$$

where $B_{t,n}$ is the price of a bond at t paying a unit of the numeraire good at period $t+n$, with the usual boundary condition that $B_{t,0} = 1$.

The futures price of the oil good, y, is defined as follows. $F_{t,n}$ is the price agreed to in period t for delivery n period hence. Futures prices satisfy

$$\begin{aligned} 0 &= E_t[m_{t+1}(F_{t+1,n-1} - F_{t,n})] \\ \Rightarrow F_{t,n} &= (B_{t,1})^{-1} E_t[m_{t+1} F_{t+1,n-1}], \end{aligned} \tag{24}$$

with the boundary condition $F_{t,0} = P_t$.

3.4 Portfolios

One interesting feature of a multi-agent model is that we can look directly at the role of financial markets in implementing the Pareto optimal allocations, by defining a set of tradeable financial assets that dynamically complete the market. In particular, we are interested in how oil futures are traded in such a setting. We defer that specific question to our numerical example of the model, since we lack analytical expressions for futures prices.

However, we can look analytically at how “equity” claims can implement the optimal allocations. Recall that C_t^x is the value of a claim to the stream of numeraire good and C_t^y is the value of the claim to the stream of the oil good. We think of these as (unlevered) claims to equity in the numeraire and oil sectors, and normalize the shares outstanding in each sector to one. Suppose these were traded claims in the economy. Is there a portfolio of $\phi^x_{1,t}$ shares in numeraire and $\phi^y_{1,t}$ shares in oil that implement optimal consumption for agent one, with $1-\phi^x_{1,t}$ and $1-\phi^y_{1,t}$ shares optimal for agent two? This amounts to replicating the optimal wealth process for each agent using the equity claims. It turns out this is easy to solve. The agents' budget constraint are:

$$
\begin{aligned}
C_{1,t} &= \phi^x_{1,t} C^x_t + \phi^y_{1,t} C^y_t \\
C_{2,t} &= \phi^x_{2,t} C^x_t + \phi^y_{2,t} C^y_t
\end{aligned}
$$

Substitute in the definition of the aggregate value of the numeraire sector in equation (20) and oil sector in equation (21). The key here is that for each agent, the value of the oil consumption stream is proportional to the value of the numeraire stream, i.e.,

$$
\frac{C^y_{1,t}}{C^x_{1,t}} = \frac{\gamma_1}{1-\gamma_1} \quad , \quad \frac{C^y_{2,t}}{C^x_{2,t}} = \frac{\gamma_2}{1-\gamma_2}.
$$

This all implies, for agent one:

$$
C^x_{1,t} = \frac{(1-\gamma_1)\left((-\gamma_2 C^x_t + (1-\gamma_2)C^y_t\right)}{\gamma_1 - \gamma_2} \tag{25}
$$

and

$$
\phi^x_{1,t} = \frac{-\gamma_2}{\gamma_1 - \gamma_2} \tag{26}
$$

$$
\phi^y_{1,t} = \frac{1-\gamma_2}{\gamma_1 - \gamma_2}. \tag{27}
$$

Equivalent results hold for agent two. Note that shareholdings are constant. As we will see in the numerical section in a moment, optimal consumption for the two agents, and the implied prices and asset returns, have many interesting dynamic properties. These properties depend on the interaction of preferences over goods, intertemporal substitution, and risk-aversion over states. However the homogeneity of the preference structure means that portfolio policies are "buy and hold" if equity claims to each sector are tradeable. Furthermore, equity portfolios depend only on the agents' relative preference for oil consumption. These results caution against taking portfolio holdings as an adequate summary of agent attitudes towards risk.

While this result is interesting, perhaps it is not all that practical. The equity positions implied by Equation (27) are extreme for reasonable values of γ_1 and γ_2, for example if agents have relatively similar preferences over goods. And of course not all claims to oil production revenues are financed through publicly traded equity. In the numerical section, next, we look at portfolio policies that implement optimal consumption using oil futures contracts. This also gives us a perspective on open-interest dynamics.

4 Numerical Example

The recursive Pareto problem is hard to characterize analytically, so we look at a numerical example. The goal of the example is to capture enough of the salient features of the data to be quantitatively informative, while remaining simple enough to inspect results. In single agent endowment models, obtaining a reasonable equity premium requires either highly persistent risks to consumption growth as in Bansal and Yaron (2004), or rare disaster-like risk as in Barro (2009). Time-variation in risk premiums requires stochastic volatility (e.g., Bansal and Yaron (2004)) or variation in the likelihood of disaster (e.g, Wachter (2013)). For simplicity we assume a four state Markov process for annual growth, described in Table 5, with growth in x as $f(s)$ and growth in y as $g(s)$. Notice that the process has a disaster-like risk in states one and four. In our two-agent economy, which of those states is the bigger concern will drive risk premiums. Endogenous variation in the wealth distribution is an additional source of time-variation in the risk premium, by weighting aggregate concern towards one disaster state or the other.

The parameters in Table 5 imply that the numeraire (x) and oil (y) processes have unconditional mean growth rates of 2% per year. Unconditional standard deviations of numeraire and oil consumption growth are 3% and 6%, respectively. The unconditional correlation in growth rates is positive, at 0.44. Historically, oil consumption represents approximately 4% of US GDP (Hamilton (2008)). In our model this characteristic is governed chiefly by the choice of goods aggregation parameters. All preference parameters are listed in Table 6. In the Cobb-Douglas aggregator we use $\gamma_1 = 0.03$ and $\gamma_2 = 0.06$. This gives agent two a greater preference for oil consumption than agent one, while keeping oil within a plausible range of the 4% historical average. Risk aversion (α_i) and intertemporal substitution (ρ_i) parameters address the risk premiums on equity and oil futures. We have selected these parameters to maintain a plausibly high but variable equity risk premium, while allowing for more dramatic variation in the oil futures risk premium. Specifically, a single agent economy with just agent one would have an equity risk premium of about 2% per year, whereas with just agent two the equity premium is about 4%. Not surprisingly, our simple four state Markov process is too coarse a description to completely capture all aspects of equity returns.[10] Notice that agent one has a higher risk aversion coefficient. This does not directly translate to higher "risk aversion" (and a higher equity risk premium) since the preferences include, recursively, the sensitivity to oil risk. In some sense, agent two faces "more risk" from the higher exposure to the oil-good. This interplay is helpful for understanding the numerical results that follow.

Since the model is stationary in growth, initial levels of x_0 and y_0 do not impact returns or risk premia. The level of the spot price, however, depends on the ratio of oil and numeraire, as in equation (14). We choose the (arbitrary and innocuous) initial level of x_0 and y_0 so that spot prices are at a familiar per-barrel level (e.g., \$60/bbl). However, since the endowment levels of x_t and y_t are not cointegrated, the spot price of oil in this specification is non-stationary. Since we focus on returns in what follows this plays no

[10]We abstract from many of the usual model characteristics such as dividends being distinct from consumption, leverage, and so on.

role.[11] Whether or not the spot price of oil is in fact stationary is an interesting question that involves the interplay of technological progress affecting oil demand and supply. A gallon of gasoline yields more miles in 2015 than 1971 and changes in shale oil (and gas) technology have had a large impact on supply. This is a central issue in resource economics that more carefully considers the long run implications of "peak oil," is an addressed in Ready (2014) and David (2015), but is a side issue in our setting.

With different parameter values, our model could easily be used to study markets other than oil. For example, its basic structure is similar to the international finance model in Colacito and Croce (2013), which features two agents, representing countries, with heterogeneous preferences over foreign and domestic consumption goods. By restricting the form of preference heterogeneity and the growth process, their model takes on useful features such as a stationary wealth distribution, as proved in Colacito and Croce (2014). But modeling the oil market requires more general preference heterogeneity than these restrictions allow. For example, we cannot assume that agents have symmetrical goods preferences ($\gamma_1 = 1 - \gamma_2$), because it would unrealistically imply oil is greater than 50% of consumption expenditures for one of the agents. Neither can we assume $\alpha_1 = \alpha_2$ and $\rho_1 = \rho_2$, because we could not obtain the variety of risk premia we seek.

For the oil futures risk premium, our goal is to generate variation that is in a range consistent with recent data. Hamilton and Wu (2014), for example, suggests an annualized risk premium to a long position in the 8-week contract of around 4% in 1990, falling to around -5% in 2011. Our endowment and preference parameters generate an unconditional oil futures risk premium of about 3% in an economy populated solely by agent one, and a -2% unconditional risk premium under an economy of only agent two. We can then investigate

[11] It is easy to specify a growth process with a stationary oil price by specifying the growth dynamics of the numeraire and for the ratio of oil-to-numeraire. While the results are broadly similar in such a setup it is harder to generate sizable risk premiums for oil. Roughly, imposing the assumption that oil and the general economy are co-integrated implies that "oil risk" is smaller. This is analogous to why our four state Markov model looks disaster-like rather than long-run-risk. The long run risk aspect is hard to capture in a small number of states.

how the dynamic risk-sharing properties of the model might endogenously generate some of the change we see in risk premiums over the 1990 to 2015 period, without any structural breaks.

4.1 Wealth and consumption

With the growth process in Table 5 and the preference parameters in Table 6, we solve the dynamic recursive Pareto problem outlined in (12). The exogenous Markov growth is characterized by its current state, s. The key endogenous state variable in the solution of the Pareto problem is the promised continuation utility level for agent two, $\hat{V}$. Since $\hat{V}$ is bounded and positive, we normalize this state variable to the $[0, 1]$ interval.[12]

Figure 4 plots the oil consumption share, the numeraire good consumption share, and the wealth share as a function of the endogenous promised utility level. The figure integrates across the exogenous growth state, s, at the unconditional stationary probabilities. In this example, the shares of aggregate wealth (bottom panel) are nearly linear in $\hat{V}$. So a value of $\hat{V} = 0.25$ corresponds to agent two owning roughly 25% of aggregate wealth. The curvature in the top panel of Figure 4 highlights the two agents' differing preference for oil: agent two has a stronger preference for oil, and so tends to get a disproportionate share.

Risk sharing models are often non-stationary, in the sense that the long horizon has one agent dominating the economy; see, e.g., Yan (2008) and Anderson (2005). In our numerical example, $\hat{V}$ drifts towards one. We do not derive analytical survival or extinction conditions for the general case with heterogeneous Epstein-Zin preferences and multiple goods, and to

[12] Although an agent's utility may grow as the economy expands, recall that $\hat{V}_t = \frac{V_t}{A_2(x_t,y_t)}$ is bounded for any x_t and y_t. However the domain of $\hat{V}$ is determined in equilibrium based on the model parameters. Its minimum value is 0, and its maximum $\hat{V}_{\max}$ corresponds to the case where agent two consumes the aggregate output of each good, i.e. $c^x_{2,t} = x_t$, $c^y_{2,t} = y_t$ in perpetuity. This lets us normalize by dividing $\hat{V}$ by $\hat{V}_{\max}$, to obtain a state variable between zero and 1. To be precise, since $\hat{V}_{\max}$ depends upon the current growth state, we normalize $\hat{V}$ by its conditional maximum.

our knowledge such results are not presented in prior work.[13] Although simulations suggest that agent two will dominate the economy at long-horizons, the dynamics of $\hat{V}$ are slow and low frequency relative to the exogenous Markov growth process. Figure 5 shows the density of $\hat{V}$ over long horizons from of 10 and 100 years, starting at an initial value of $\hat{V}_0 = 0.05$. As we discuss later, we choose the low initial $\hat{V}_0$ to illustrate how a drift in wealth share may produce apparent trends in open interest, the spot price, and the futures risk-premium.

4.2 Risk premia

Prices and conditional expected returns and risk premiums depend on the exogenous Markov growth state, s and the endogenous state variable $\hat{V}$, effectively the wealth share of agent two. We use our numerical example to look at behavior conditional on each of the exogenous and endogenous state variables. Figure 6 plots mean oil price and returns conditional on the wealth share $\hat{V}$. They are "mean" values in the sense that we integrate out the exogenous state s at its stationary (unconditional) likelihoods, holding constant the wealth share. Notice that the (mean) oil spot price is increasing in the wealth share of agent two. This reflects, of course, agent two's greater preference for the oil good. The risk premium in oil futures is positive for low levels of $\hat{V}$ and is negative for higher levels $\hat{V}$. That is, when agent one has a high wealth share (low $\hat{V}$), oil futures have a positive risk premium and when agent two is more wealthy the risk premium is negative. Interestingly, asset returns for the numeraire good, the risk free rate and the equity risk premium, are not monotonic in the wealth share. Near a wealth share of approximately 0.3, the equity premium is highest, the risk free rate is at its lowest, and the oil risk premium is near zero.

Before digging into the mechanism that is driving these results, we characterize our example along the same lines as the returns data, starting with contracts of differing matu-

[13] Colacito and Croce (2014) present an example of a stationary two-agent economy with Epstein-Zin preferences and two goods, but prove stationarity under several assumptions not imposed in our study, such as a symmetrical growth process, and preference heterogeneity limited to symmetrical differences in the goods parameter value.

rity. Figure 7 presents the term-structure of oil futures risk premiums. Notice that the risk premiums are lower when agent two has a larger wealth share (larger $\hat{V}$) at all horizons, as we saw in Figure 6 for the near-term contract. The fact that risk premiums are declining in maturity is interesting. In an economy dominated by agent one (low $\hat{V}$), the largest risk premium is positive and at the short end. In contrast, in the economy with a higher wealth share for agent two (high $\hat{V}$), the sizable risk premium is negative and at the long end. Hitzemann (2015) emphasizes oil-sector exposure to long run risk, which is not included in our model, as a mechanism that generates a risk premium increasing in maturity. Although disaster risk and long run risk have not yet been studied in a combined oil economy framework, the natural supposition is that the two sources of risk would offset each other's effects on the term structure of risk premiums. Historical average excess returns show a relatively flat term structure overall, particularly for contracts out 12 months or more, as illustrated in Figure 2.

Turning to the conditional risk premiums, Figure 8 plots the oil risk premium on the short-horizon two year contract, conditional on both the endogenous wealth share, $\hat{V}$, and the endogenous growth state, s. This shows that the risk premium is low in state $s = 1$, when oil is growing relatively scarce, and vice versa for state $s = 4$, when oil is growing relatively abundant. We cannot directly observe analogous growth states in the data. However we can condition on the slope of the futures curve in the model. Table 7 shows risk premiums conditional on the slope of the futures curve in the model, analogous to Figure 2 for the data. Note that the risk premium is conditionally higher, and generally positive, when the futures slope is negative.

Table 8 uses the current slope to predict excess futures returns, via regressions equivalent to those performed on the data. Regression coefficients are computed via simulation, with an initial value of $\hat{V}$ as specified in the table. We report average coefficients from 10,000 paths of 100 periods each. As in the data, in Table 1, the coefficient on the slope is negative for all contracts, so a downward sloping futures curve implies higher expected excess futures returns. However t-statistics from the model regressions are not significant at conventional levels for larger values of $\hat{V}$. That is, the slope becomes a less effective predictor of excess returns as agent 2 becomes wealthier. The futures curve is also less frequently downward sloping for larger $\hat{V}$, e.g., the slope is negative about 80% of the time in for the smallest $\hat{V}$ versus only 6% of the time for the largest $\hat{V}$. These results illustrate that apparent empirical regularities in the futures market may be fragile if trade reflects heterogeneous preferences, because wealth may shift between types over time.

In the data, we also examined the cross-market relationship between crude oil futures and bonds. Figure 10 shows the term structure of real interest rates conditional on the wealth share, $\hat{V}$. For most values of $\hat{V}$ the term structure is relatively flat, with slightly downward sloping rates around 1.25% for $\hat{V} = 0.35$, and slightly upward sloping rates around 2.25% for $\hat{V} = 0.95$. The exception is $\hat{V} = 0.05$, which is steeply downward sloping. Ang, Bekaert, and Wei (2008) estimate that unconditional real interest rates are relatively flat around 1.3%, with some regimes in which the real rate curve is steeply downward sloping. They attribute the upward sloping nominal term structure to an inflation risk premium, which of course we cannot replicate without adding money to our model. However we can still see how (real) bond risk premia in our model relate to the slope of the crude oil futures curve.

Recall that Table 2 and Table 3 show how the Treasury bond risk premium correlated with the oil risk premium, as measured by the futures slope. Table 10 replicates the regression of annual excess bond returns on the futures slope, reporting average coefficients for 10,000 paths of 100 periods each. We do see dependence on the futures slope, but with the opposite correlation to the data. In the data, particularly the 2004 period, a negative slope was indicative of a larger bond risk premium. Possibly the negative coefficient in the data results from the futures curve predicting the inflation risk premium. One aspect the model example seems to capture is that the degree of correlation – as seen in Table 9 by the difference between the term premium conditional on a positive or negative oil futures curve – varies over time with the endogenous wealth share state. Notice in the table that around

$\hat{V} = 0.35$ the relationship changes from a higher term premium with positive slope to a higher term premium with negative slope. This illustrates that the relationship between risk premia across markets may not be robust over time, even in the absence of structural breaks.

4.3 Why?

The key driver in the numerical example is that, on average, agent one views a long position in oil as risky and requires a positive risk premium, whereas agent two views the long position in oil as a hedge and accepts the negative risk premium as insurance. Why is that? Our four state growth process in Table 5 has the feel of a "disaster" model. States 1 and 4 are the bad-outcome states. State 1 has low numeraire growth, very low oil growth, and a rising price of oil. This state loosely resembles a recession induced by an oil-supply shock. State 4 has very low numeraire growth, plentiful oil growth, and a falling price of oil. This is reminiscent of a deep recession whose roots do not lie in oil. The two agents in the example rank these states differently.

We can see the agents' ranking of the growth states by looking at the state price densities. Recall that optimality aligns the two agents' marginal rates of substitution, state-by-state, as in equation (13). The state price density, $m_{t+1,t}$, is for moving to growth state s_{t+1} and wealth share $\hat{V}_{t+1}$ given the current s_t and wealth share $\hat{V}_t$. Figure 9 captures the salient informaton in the state price density by plotting the average $m_{t+1,t}$ for each of the four growth states s_{t+1}, integrating out the current growth state.[14] The figure lets us look at state prices for the four growth states while varying the wealth share of agent two; the left of the graph is an economy of mostly agent one and the right is an economy of mostly agent two.

Denote the state prices shown in Figure 9 as M_s. The two lower lines of Figure 9,

[14] The figure looks very similar across various alternative averaging.

M_2 and M_3, are both small. The two agents view these states similarly and as equally bountiful; these prices vary little as the wealth share moves. The upper lines are for states 1 and 4. Both of these states are "bad," and weigh heavily on asset returns. For agent one, the low-volume-oil consumer, state 4 is the worst: for $\hat{V}$ near 0, $M_4 > M_1$. In state 4 the spot price of oil decreases, hence a long position in oil has a low payoff. Therefore, to agent one, oil futures are risky and command a positive risk premium. For agent two, the agent with the higher dependence on oil, the low growth of oil in state 1 dominates: for $\hat{V}$ near 1, $M_1 > M_4$. In state 1 the spot price of oil increases. To agent two, a long position in oil futures has a large payoff precisely when the marginal rate of substitution is high. This hedge, in equilibrium, produces the negative risk premium to a long position in oil.

For ranges of the wealth share that are not near the boundary, we can see the impact of risk sharing. For $\hat{V}$ around 0.35, the risk premium on oil, at the short end, is close to zero; see Figures 6 and 7. In our example, this is where "hedgers" and "speculators" have equal demand. Looking back at Figure 6, this is near the region where the risk free rate is particularly low and the equity risk premium is at its highest. Here, it seems, with the oil risk sharing as large as possible (i.e. a zero oil risk premium), the scope for risk sharing in the numeraire good is smaller, and hence the two numeraire claims reflect this. The risk free rate is low, and the equity premium is high. The discussion here focuses on one-period gambles, as captured in Figure 9. Risk premiums are more complicated a longer horizons. Both agents view the long-oil futures position as more of a hedge at longer horizons. The risk premiums in Figure 7 are generally downward sloping.

4.4 Portfolios and open interest

As we saw in Section 3.3, when claims to aggregate consumption of the numeraire and oil are traded in financial markets, then the agents can implement their optimal consumption plans with a constant buy-and-hold portfolio. In our example, agent one would hold 2 shares of the claim to aggregate numeraire consumption (akin to a broad equity portfolio excluding the oil sector), and roughly −32.3 shares of a claim to oil consumption (say shares in Exxon, Chevron, and so on). The aggregate supply of each of these two assets is normalized to one. So agent two's portfolio is short one share of the equity claim and is long 33.3 shares of the oil claim.[15] Neither of these portfolios is particularly practical. In reality investors are unable to trade a claim to aggregate consumption of oil. Much of world oil production is in state-owned enterprises (e.g., Saudi Aramco, PDVSA) that are not publicly traded. In practice, we see trade in futures contracts. And, as noted earlier, a large increase in trading volume occurred post 2004. We can implement of our Pareto-optimal allocations using futures contracts to (dynamically) complete the market. With four growth states, we can do this with four assets. Here, we use a claim to aggregate consumption of both goods C (call this "equity"), the one-period risk free bond B_1, and two fully-collateralized oil futures contracts. We use the one and two year horizon contracts (labeled F_1 and F_2, respectively).

Figure 11 shows each agent's optimal portfolios for a range of wealth shares of agent two (the endogenous state-variable $\hat{V}$). Market clearing implies that agent two's holdings of futures contracts are the mirror image of agent one's: the bond and the futures contracts are in zero net-supply, whereas equity has a net supply of one. The equity claim, denoted C in Figure 11, mostly reflects the relative wealths of the two agents. Recall that it is agent two who most desires to hedge the oil-price risk. Hence, agent two's portfolio is long F_1 – the futures contract that is most sensitive to the oil spot price. It is interesting that getting the right sensitivity to the price of oil requires a long position in F_1 and a short position in the longer-dated contract F_2: a calendar spread. So the overall hedging motive of a trader could easily be misinterpreted based on the direction of trade in a single contract.

There has been extensive policy debate around increases in open interest in the oil

[15]The portfolios specified are nevertheless equivalent to the dominant agent holding one share of each stock when $V \to 0$ or $V \to 1$.

futures market. Figure 12 plots open interest, defined here as the absolute value of agent one's position. Not surprisingly, open interest is small when the economy is dominated by one agent ($\hat{V}$ near 0 or 1). Interestingly, the open interest is largest near $\hat{V} = 0.35$. This corresponds to where the oil risk premium is zero – the point where "hedgers" and "speculators" are equally balanced. Recall that the price of oil is increasing in the share of wealth for agent two, as shown in the right panel of Figure 12. So in this example, an increase in open interest and an increase in the oil price occur as $\hat{V}$ moves, say, from 0.05 to 0.35.

In the data, we also saw in Table 4 that open interest is positively correlated with the futures slope, which is an indicator of the risk premium. However we noted that open interest does not directly predict excess futures returns at a statistically significant level. Table 11 shows that open interest is also significantly, but imperfectly, positively correlated with the futures slope in the model. Variation in open interest partially reflects the state of nature s, but as we see in Figure 12, open interest also varies nonmonotonically with $\hat{V}$. The futures risk premium varies with both s and $\hat{V}$. As a result, based on simulated regressions, open interest does not directly predict excess futures returns at a statistically significant level in the model either.

4.5 Wealth Dynamics

As we noted previously while examining Figure 5, the wealth share of agent two, $\hat{V}$ has positive drift in our example. Quantitatively, what are the effects of this drift, and how do they compare to data? Figure 13 shows an average path (across 10,000 simulations) for agent 2 wealth share ($\hat{V}$), the spot price, the futures risk premium, and open interest over a 50 year horizon. Open interest and the risk premium use the one-year (nearest) contract. As the drift in wealth share is relatively slow, we choose the initial $\hat{V}$ to be in a region where the economy is more sensitive to changes in the wealth distribution. We initialize the simulation at $\hat{V} = 0.05$. This is an economy dominated by agent one. Agent one has a lower preference for oil, so the initial spot price is low. Also, since $\hat{V} = 0.05$ has one dominant agent, open interest is low. As the economy evolves, $\hat{V}$ increases and agent two has a larger role. Over the 50 year horizon, the expected oil price roughly doubles, as does expected open interest as a fraction of total wealth. The risk premium falls by about 1/3, a change consistent in direction with split sample averages for near-to-maturity contracts, although not for long-dated contracts.[16]

The results suggest that, at least over a horizon of decades, the wealth distribution is an important driver of the amount of trade in futures, and the futures risk premium. And indeed there would be changes in goods market prices coinciding with financial market changes, although the magnitude of these must be interpreted with care. Most of the expected increase in the spot price results from the correlated growth process for the oil and non-oil goods: the increase in spot price attributable to the change in wealth distribution is about 8%. So the run up in oil prices before 2008, for example, would have to stem from the relative growth in aggregate consumption of oil, not from a shift in wealth alone.

5 Conclusion

We have focused on heterogeneous attitudes towards oil risk as an important driver of oil risk premium dynamics, price dynamics, and trade evident in the data. To attack this question, we look at the frictionless consumption sharing problem for two agents with different attitudes towards consumption risk and, specifically, the oil-component of consumption. The solution lets us look at consumption and wealth paths and the implications for risk premiums. Changes in market behaviour follow shifts in the wealth distribution, which are endogenous and Pareto optimal. In a numerical example, we generate in expectation

[16] In our example, long-dated contracts would also see a decline in risk premium on average in the simulation.

rising oil prices, decreasing risk premiums, and increasing open interest. However the example suggests that consumption or wealth share dynamics are unlikely to account for rapid variation oil futures market dynamics, except perhaps for open interest.

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Table 1: **Regression of monthly excess futures return on slope of futures curve: 1990-2016**

Contract	Constant	Slope	R^2
CL 2	-0.049	-5.534	0.006
	(-0.075)	(-1.099)	
CL 3	0.165	-5.645	0.007
	(0.273)	(-1.327)	
CL 6	0.338	-5.489	0.008
	(0.641)	(-1.411)	
CL 12	0.327	-5.912	0.014
	(0.742)	(-1.719)	
CL 18	0.212	-6.261	0.021
	(0.540)	(-2.026)	
CL 24	0.165	-7.183	0.031
	(0.448)	(-2.909)	
CL 36	0.182	-7.728	0.042
	(0.464)	(-3.410)	
CL 48	0.220	-7.505	0.039
	(0.555)	(-3.253)	
CL 60	0.311	-8.098	0.045
	(0.792)	(-3.623)	

Excess holding period returns are monthly log changes in contract value $r^n_{oil,t+1} - r_{f,t+1} \approx \log F_{t+1,n-1} - \log F_{t,n}$., and the slope is $\log F_{t,18} - \log F_{t,1}$. Data is NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month from Jan. 1990 through April 2016. T-statistics in parenthesis use HAC standard errors with $N^{1/3}$ lags, where N is the number of observations.

Table 2: **Regression of excess bond returns on slope of futures curve and Cochrane-Piazzesi factor: 1990-2003**

Bond Index	Constant	Slope	CP	R^2
1 Year Bond	1.253	-3.343		0.130
	(5.679)	(-2.483)		
	1.233		0.212	0.063
	(4.128)		(1.663)	
	0.925	-3.990	0.286	0.241
	(3.846)	(-3.183)	(2.810)	
2 Year Bond	1.881	-6.904		0.149
	(4.208)	(-2.838)		
	1.632		0.642	0.156
	(2.929)		(2.761)	
	0.959	-8.725	0.804	0.384
	(2.196)	(-4.005)	(4.654)	
5 Year Bond	2.902	-13.883		0.149
	(3.142)	(-2.972)		
	2.065		1.621	0.248
	(2.013)		(3.916)	
	0.653	-18.328	1.963	0.497
	(0.833)	(-4.669)	(6.944)	
10 Year Bond	3.244	-16.922		0.105
	(2.257)	(-2.263)		
	1.444		2.744	0.334
	(1.095)		(5.482)	
	-0.417	-24.155	3.194	0.538
	(-0.362)	(-4.616)	(9.245)	
20 Year Bond	4.488	-22.002		0.134
	(2.803)	(-2.635)		
	2.778		2.947	0.293
	(1.833)		(4.939)	
	0.472	-29.940	3.505	0.531
	(0.378)	(-5.336)	(8.585)	
30 Year Bond	3.366	-24.360		0.108
	(1.651)	(-2.244)		
	1.278		3.455	0.265
	(0.684)		(4.804)	
	-1.310	-33.602	4.081	0.463
	(-0.741)	(-4.219)	(7.459)	

Monthly continuously compounded CRSP bond index returns in excess of the 30-day T-bill rate are summed to obtain annual returns. The slope is $\log F_{t,18} - \log F_{t,1}$, and CP is the factor from Cochrane and Piazzesi (2005). Data for NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month from Jan. 1990 through Dec. 2003. T-statistics in parenthesis use HAC standard errors with $N^{1/3}$ lags, where N is the number of observations.

Table 3: **Regression of excess bond returns on slope of futures curve: 1990-2016**

Bond Index	Constant	Slope	R^2
1 Year Bond	0.877	-2.081	0.047
	(5.386)	(-1.890)	
2 Year Bond	1.595	-3.245	0.035
	(5.136)	(-1.648)	
5 Year Bond	2.952	-7.168	0.044
	(4.718)	(-2.025)	
7 Year Bond	3.793	-8.969	0.044
	(4.809)	(-2.058)	
10 Year Bond	3.757	-10.096	0.041
	(4.045)	(-1.939)	
20 Year Bond	5.298	-17.840	0.071
	(4.487)	(-2.777)	
30 Year Bond	4.791	-23.647	0.073
	(3.106)	(-2.610)	

Monthly continuously compounded CRSP bond index returns in excess of the 30-day T-bill rate are summed to obtain annual returns. The slope is $\log F_{t,18} - \log F_{t,1}$. Data for NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month from Jan. 1990 through Dec. 2003. T-statistics in parenthesis use HAC standard errors with $N^{1/3}$ lags, where N is the number of observations.

Table 4: **Regression of slope of futures curve on open interest: 1990-2016**

Constant	Open Interest	Commercial Imbalance	R^2
-0.025	0.243		0.102
(-1.571)	(3.434)		
-0.025		0.482	0.077
(-1.485)		(3.410)	
-0.025	0.271	0.555	0.202
(-1.655)	(3.478)	(4.260)	

CFTC data for open interest, measured as the number of contracts, is sampled on the last available day of the month, and a linear trend in logs is removed. Commercial imbalance is the detrended percentage difference between commercial long and short futures positions, also from the CFTC. The slope is $\log F_{t,18} - \log F_{t,1}$ for NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month. All data is from Jan. 1990 through April 2016. T-statistics in parenthesis use HAC standard errors with $N^{1/3}$ lags, where N is the number of observations.

Table 5: Aggregate Consumption Growth Process

$$\begin{bmatrix} s \in S \\ f(s) \\ g(s) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0.99 & 1.03 & 1.05 & 0.93 \\ 0.90 & 1.04 & 1.06 & 1.07 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.05 & 0.05 \\ 0.05 & 0.18 & 0.72 & 0.05 \\ 0.05 & 0.05 & 0.63 & 0.27 \end{bmatrix}$$

$$\bar{\pi} = \begin{bmatrix} 0.20 & 0.48 & 0.26 & 0.06 \end{bmatrix}$$

Growth process characteristics. The first matrix shows possible growth outcomes for the numeraire ($f(s)$) and oil ($g(s)$) for each growth state (s). In matrix π, entry $\pi_{i,j}$ is the probability of transitioning from current growth state i to next period state j. The stationary (long-run) probability of being in a given growth state is shown in $\bar{\pi}$.

Table 6: **Parameters**

Parameter	Value	Description
α_1	-20	risk aversion, agent 1
α_2	-12.6	risk aversion, agent 2
ρ_1	-1.12	intertemporal substitution, agent 1
ρ_2	0.754	intertemporal substitution, agent 2
γ_1	0.03	oil preference, agent 1
γ_2	0.06	oil preference, agent 2
β	0.96	impatience, agents 1,2
x_0	1333	initial aggregate consumption, numeraire
y_0	1	initial aggregate consumption, oil

Preference parameters and initial aggregate consumption levels used in numerical examples.

Table 7: **Model-implied expected excess returns on oil futures contracts (%)**

	Horizon (yrs)	$\hat{V} = 0.05$	$\hat{V} = 0.35$	$\hat{V} = 0.65$	$\hat{V} = 0.95$
All	1	1.969	0.201	-0.508	-0.813
	2	2.556	-0.022	-1.487	-2.187
	3	2.237	-0.815	-2.606	-3.464
	4	1.726	-1.608	-3.541	-4.457
	5	1.238	-2.250	-4.236	-5.168
	6	0.833	-2.725	-4.724	-5.655
	7	0.515	-3.059	-5.053	-5.980
Slope +	1	-0.367	-1.482	-0.802	-1.061
	2	-1.050	-2.611	-1.953	-2.568
	3	-1.890	-3.659	-3.179	-3.923
	4	-2.655	-4.496	-4.183	-4.964
	5	-3.288	-5.121	-4.924	-5.706
	6	-3.789	-5.572	-5.442	-6.213
	7	-4.176	-5.890	-5.792	-6.551
Slope -	1	2.542	0.614	3.801	2.829
	2	3.441	0.613	5.358	3.404
	3	3.250	-0.117	5.795	3.272
	4	2.802	-0.899	5.873	2.991
	5	2.349	-1.545	5.850	2.734
	6	1.967	-2.026	5.811	2.540
	7	1.667	-2.364	5.780	2.403

Expected annual excess returns, in percent, computed as the average log-change in contract value. Results are shown for different horizons to maturity and values of state variable $\hat{V}$. Increasing values of $\hat{V}$ correspond to an increasing wealth share for the agent with lower risk aversion and higher preference for oil consumption. Slope is defined as the difference between the spot price and the 2-year futures price. Although the magnitude of the effect varies, a negative slope always implies a higher risk premium on holding the contract. For the two smallest values of $\hat{V}$, the slope is negative 80% of the time. For the two largest $\hat{V}$, the slope is negative 6% of the time. The stark contrast in these percentages reflects our simple, 4-state growth process.

Table 8: **Regression of annual excess futures return on slope of futures curve: model**

$\hat{V}_0$	Horizon (yrs)	β_0	β_1	R^2
	1	0.007	-0.215	0.096
		(1.307)	(-2.511)	
	2	0.008	-0.350	0.094
		(0.936)	(-2.458)	
0.05	3	0.001	-0.425	0.084
		(0.226)	(-2.249)	
	4	-0.006	-0.474	0.077
		(-0.280)	(-2.100)	
	5	-0.013	-0.508	0.073
		(-0.614)	(-2.002)	
	1	0.000	-0.199	0.074
		(0.054)	(-2.067)	
	2	-0.005	-0.310	0.069
		(-0.730)	(-1.950)	
0.35	3	-0.014	-0.367	0.062
		(-1.632)	(-1.774)	
	4	-0.023	-0.398	0.057
		(-2.259)	(-1.649)	
	5	-0.029	-0.417	0.055
		(-2.668)	(-1.568)	
	1	-0.003	-0.171	0.059
		(-0.677)	(-1.834)	
	2	-0.012	-0.252	0.051
		(-1.958)	(-1.650)	
0.65	3	-0.024	-0.289	0.045
		(-2.959)	(-1.456)	
	4	-0.033	-0.306	0.042
		(-3.623)	(-1.318)	
	5	-0.040	-0.315	0.041
		(-4.052)	(-1.230)	
	1	-0.004	-0.149	0.049
		(-1.159)	(-1.541)	
	2	-0.017	-0.209	0.041
		(-2.774)	(-1.310)	
0.95	3	-0.030	-0.233	0.037
		(-3.834)	(-1.127)	
	4	-0.040	-0.243	0.035
		(-4.509)	(-0.996)	
	5	-0.047	-0.246	0.034
		(-4.940)	(-0.910)	

On simulated data of futures returns from our model, regress:

$$\log F_{t+1,n-1} - \log F_{t,n} = \beta_0 + \beta_1 slope_t + \epsilon_{t+1},$$

where the slope is the log difference between the two-year contract and the spot price. The table shows average coefficients and R^2 for 10,000 paths of 100 periods each, conditional on an initial value of agent 2 wealth $\hat{V}_0$.

Table 9: **Model-implied term premium on bonds (%)**

	Horizon (yrs)	$\hat{V} = 0.05$	$\hat{V} = 0.35$	$\hat{V} = 0.65$	$\hat{V} = 0.95$
All	1	0.000	0.000	0.000	0.000
	2	-1.051	-0.163	0.010	0.062
	3	-1.620	-0.239	0.042	0.127
	4	-1.959	-0.272	0.083	0.188
	5	-2.177	-0.284	0.125	0.244
	6	-2.329	-0.286	0.165	0.294
	7	-2.439	-0.283	0.201	0.339
Slope +	1	0.000	0.000	0.000	0.000
	2	-0.274	-0.190	-0.018	0.038
	3	-0.510	-0.354	-0.007	0.085
	4	-0.715	-0.497	0.018	0.134
	5	-0.893	-0.621	0.048	0.180
	6	-1.047	-0.730	0.078	0.222
	7	-1.181	-0.825	0.107	0.260
Slope -	1	0.000	0.000	0.000	0.000
	2	-1.242	-0.157	0.429	0.414
	3	-1.893	-0.211	0.765	0.736
	4	-2.264	-0.217	1.033	0.990
	5	-2.493	-0.202	1.251	1.193
	6	-2.643	-0.177	1.430	1.359
	7	-2.748	-0.150	1.579	1.495

Annual yield to maturity on zero-coupon bonds in excess of the 1-period bond rate, in percent. Results are shown for different horizons to maturity and values of state variable $\hat{V}$. Increasing values of $\hat{V}$ correspond to an increasing wealth share for the agent with lower risk aversion and higher preference for oil consumption. Slope is defined as the difference between the spot price and the 2-year futures price. The connection between the slope of the oil futures curve and the bond term premium depends upon $\hat{V}$. For $\hat{V} = 0.05$, the term premium is higher given a positive slope. All other values of $\hat{V}$ imply higher term premium given a negative slope. For the two smallest values of $\hat{V}$, the slope is negative 80% of the time. For the two largest $\hat{V}$, the slope is negative 6% of the time. The stark contrast in these percentages reflects our simple, 4-state growth process.

Table 10: **Regression of annual excess bond return on slope of futures curve: model**

$\hat{V}_0$	Horizon (yrs)	β_0	β_1	R^2
	2	-0.011	0.087	0.254
		(-7.930)	(5.235)	
0.05	3	-0.014	0.124	0.225
		(-6.802)	(4.756)	
	4	-0.014	0.144	0.197
		(-5.765)	(4.261)	
	5	-0.015	0.157	0.176
		(-5.011)	(3.904)	
	2	-0.002	0.037	0.084
		(-2.092)	(2.434)	
0.35	3	-0.002	0.060	0.077
		(-1.293)	(2.229)	
	4	-0.001	0.075	0.073
		(-0.733)	(2.096)	
	5	-0.001	0.087	0.070
		(-0.364)	(2.013)	
	2	0.000	0.024	0.050
		(0.166)	(1.690)	
0.65	3	0.001	0.040	0.049
		(0.773)	(1.601)	
	4	0.002	0.050	0.048
		(1.234)	(1.533)	
	5	0.002	0.058	0.048
		(1.563)	(1.485)	
	2	0.001	0.017	0.038
		(1.357)	(1.287)	
0.95	3	0.002	0.029	0.038
		(1.924)	(1.220)	
	4	0.003	0.036	0.038
		(2.372)	(1.164)	
	5	0.004	0.041	0.037
		(2.703)	(1.122)	

On simulated data of bond returns from our model, regress:

$$r^n_{bond,t+1} - r^f_{t+1} = \beta_0 + \beta_1 slope_t + \epsilon_{t+1},$$

where the slope is the log difference between the two-year contract and the spot price. The table shows average coefficients and R^2 for 10,000 paths of 100 periods each, conditional on an initial value of agent 2 wealth $\hat{V}_0$.

Table 11: **Regression of slope of futures curve on future open interest: model**

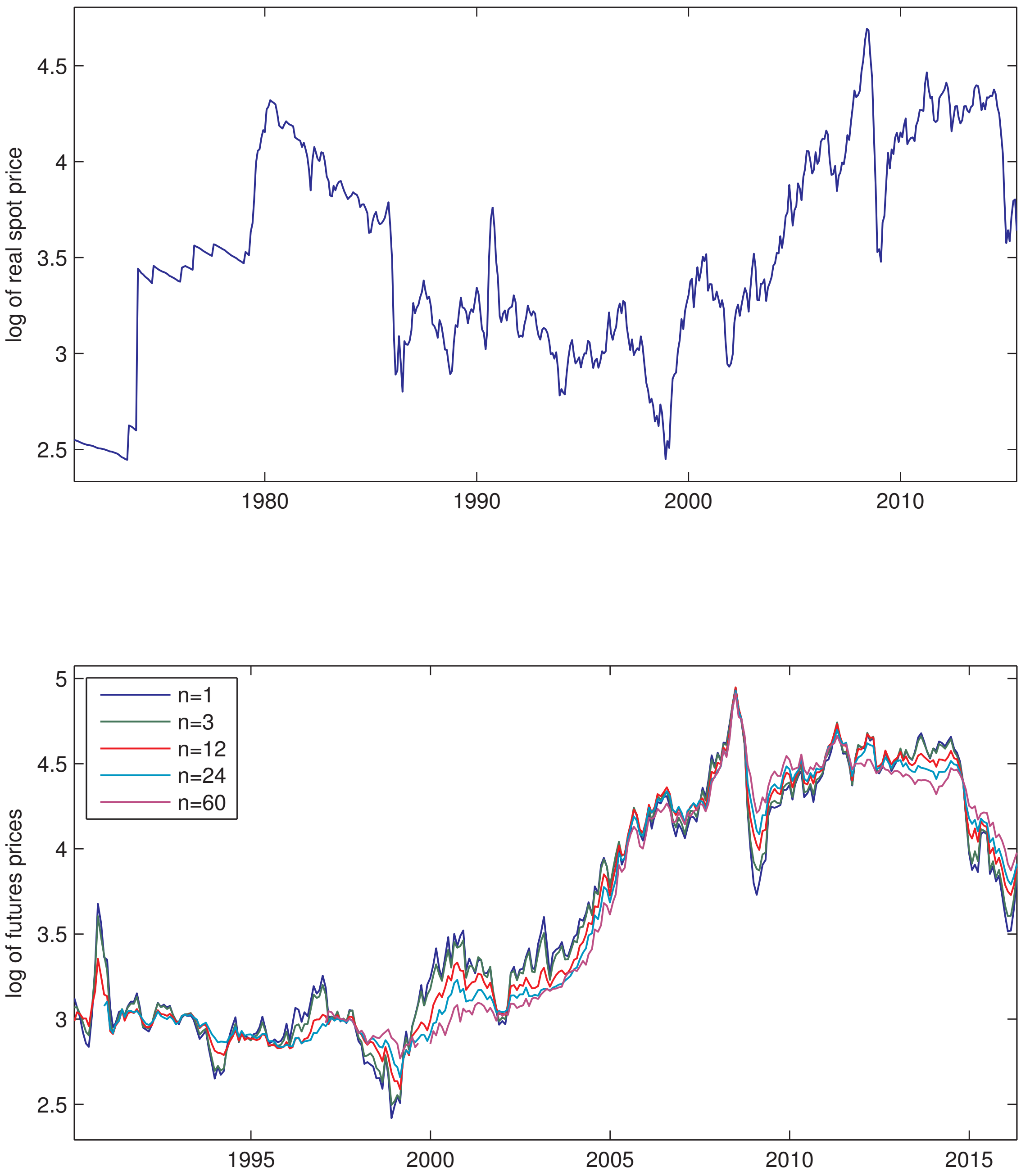
$\hat{V}_0$	β_0	β_1	R^2
0.05	-0.239	0.092	0.164
	(-3.141)	(2.723)	
0.35	-0.437	0.172	0.156
	(-2.260)	(2.333)	
0.65	-0.127	0.078	0.095
	(-1.239)	(1.514)	
0.95	0.035	0.064	0.088
	(2.845)	(1.393)	

On simulated data from our model, regress:

$$slope_t = \beta_0 + \beta_1 OI_t + \epsilon_t,$$

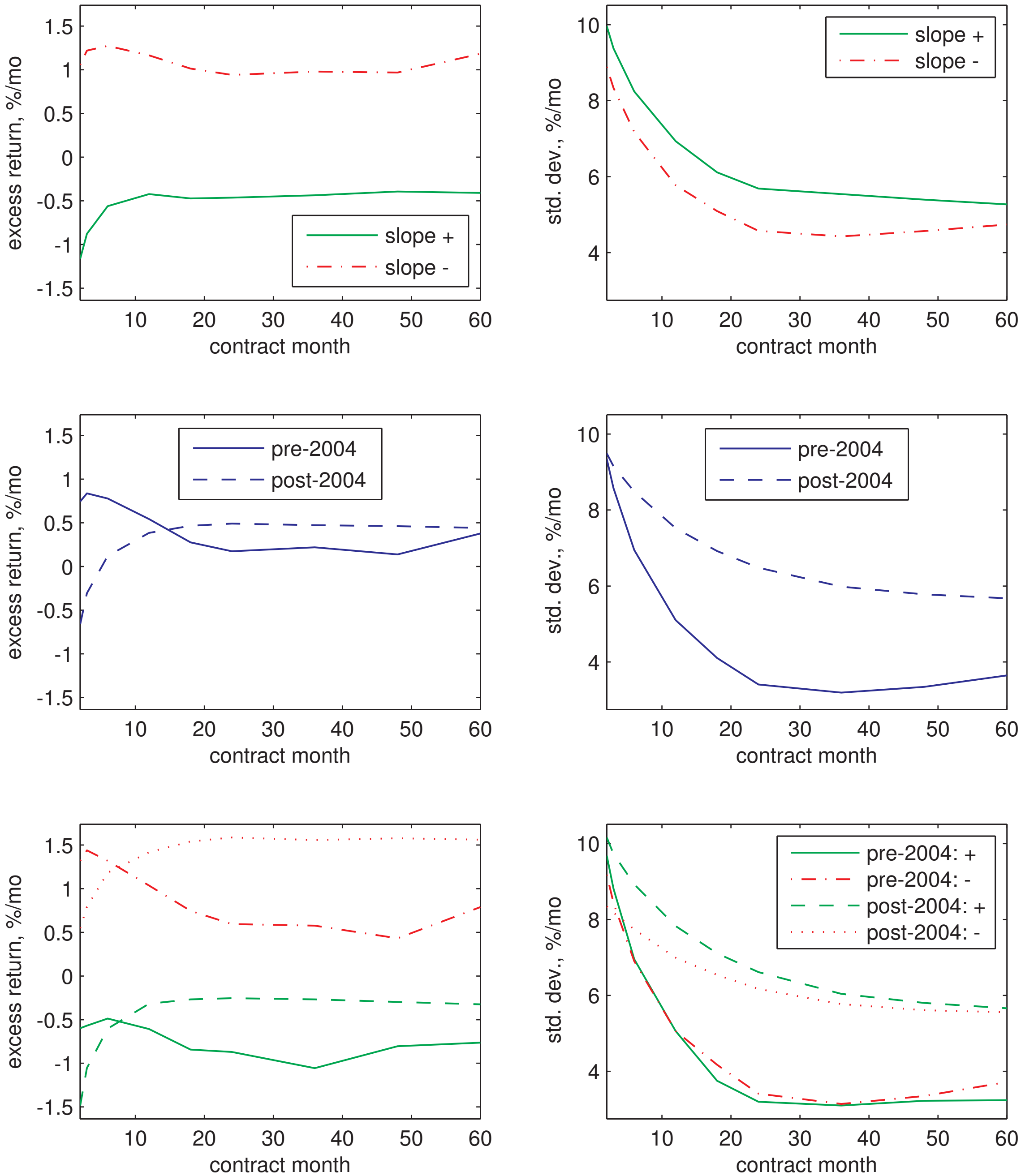
where the slope is the log difference between the two-year contract and the spot price, and open interest (OI) is the log of futures contracts outstanding. The table shows average coefficients and R^2 for 10,000 paths of 100 periods each, conditional on an initial value of agent 2 wealth $\hat{V}_0$.

Figure 1: Time series, crude oil spot and futures prices



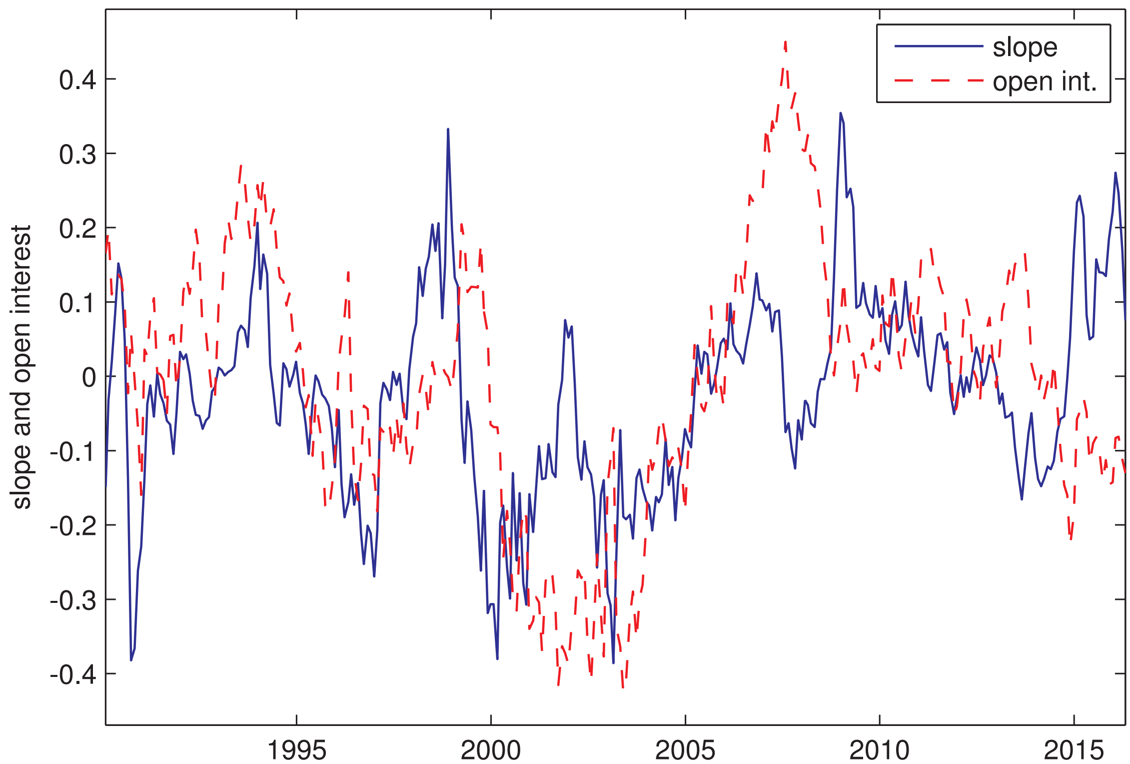
Crude oil spot prices are from the St. Louis Fed, deflated by chain-weighted price index (indexed at 2001/01), from Jan. 1971 through July 2015. Futures prices are NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month from Jan. 1990 through April 2016.

Figure 2: Crude oil futures moments by contract



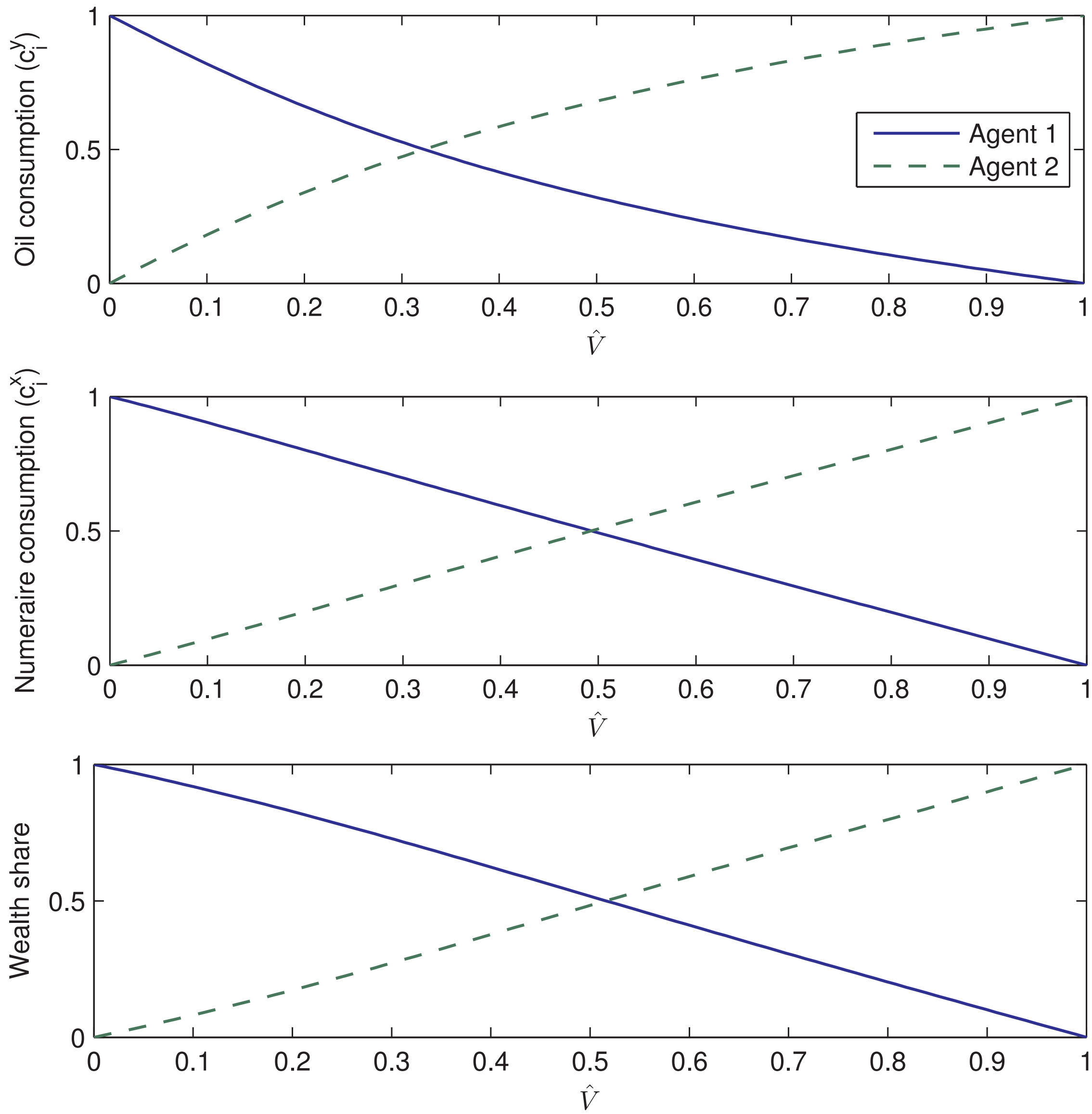
Excess holding period returns are monthly log changes in contract value $\log F_{t+1,n-1} - \log F_{t,n}$, and the slope is $\log F_{t,18} - \log F_{t,1}$. Data is NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month from Jan. 1990 through April 2016. The left column shows average excess returns, the right column shows standard deviations. Estimates are conditional on, from top to bottom, the slope, the sample period, or both. Both panels in the bottom row share the same legend.

Figure 3: Crude oil futures slope and open interest



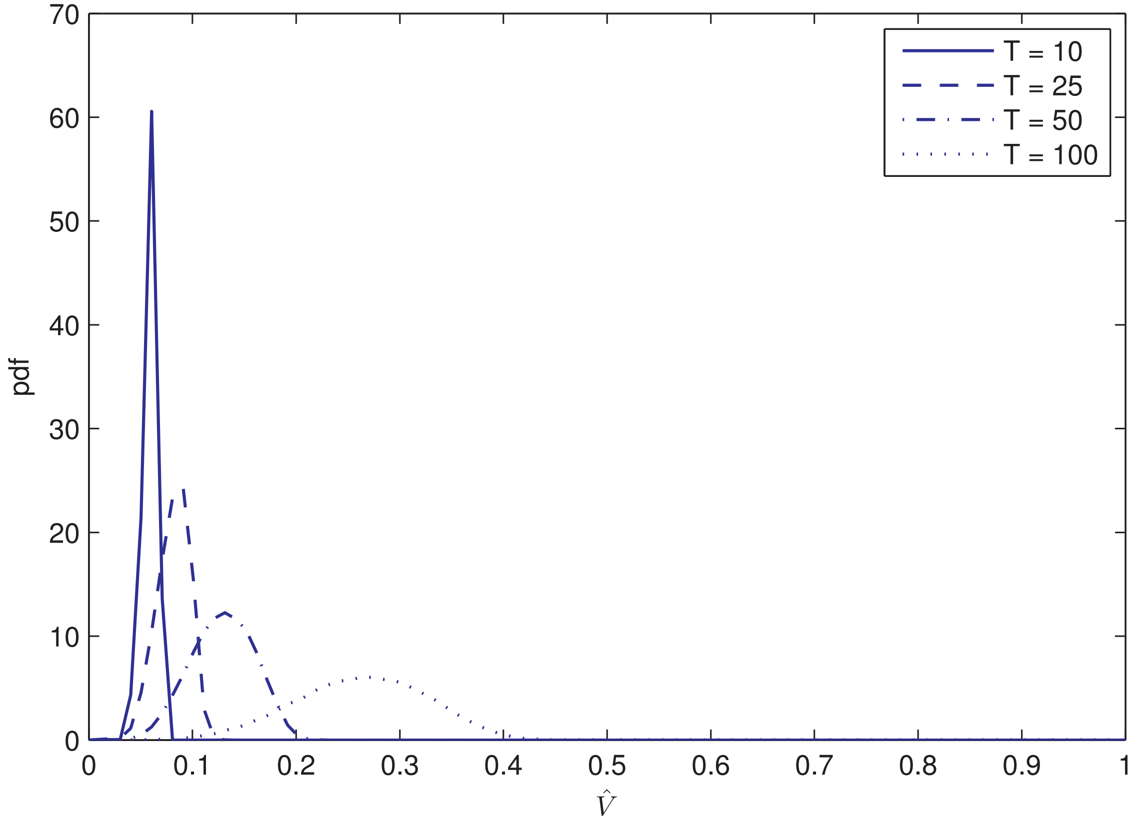
CFTC data for open interest, measured as the number of contracts, is sampled on the last available day of the month, and a linear trend in logs is removed. The slope is $\log F_{t,18} - \log F_{t,1}$ for NYMEX WTI crude oil futures, obtained from Quandl, using settle prices on the last trading day of each month.

Figure 4: Wealth and consumption shares



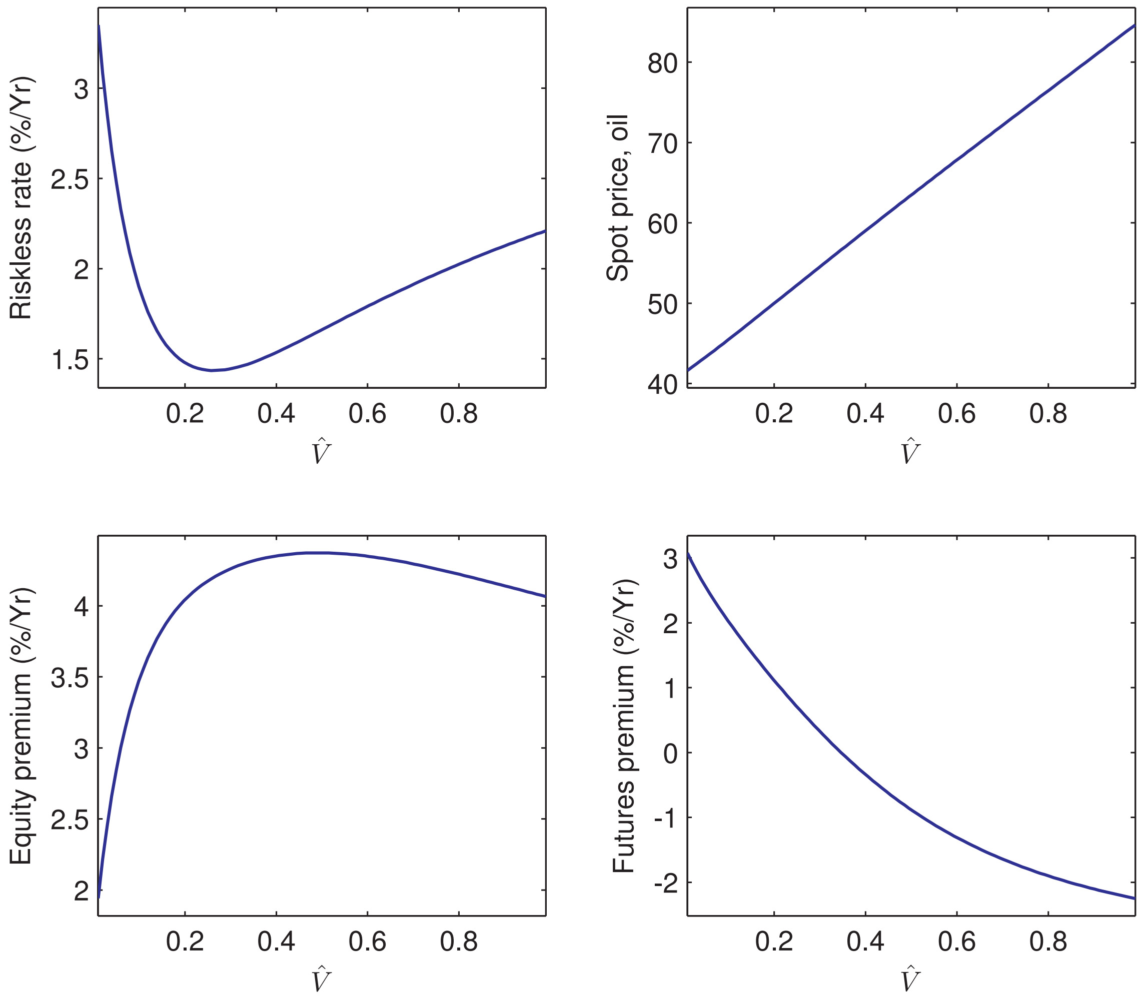
The state variable, $\hat{V}$, is the normalized promised utility to agent two. Shown are the oil consumption share (top), share numeraire consumption (middle), and wealth share (bottom). Note the shares in each graph sum to one. In each panel, the share is shown averaged across the unconditional distribution of the Markov endowment growth state s. However results conditional on a particular s are similar to the mean.

Figure 5: Probability density of $\hat{V}$



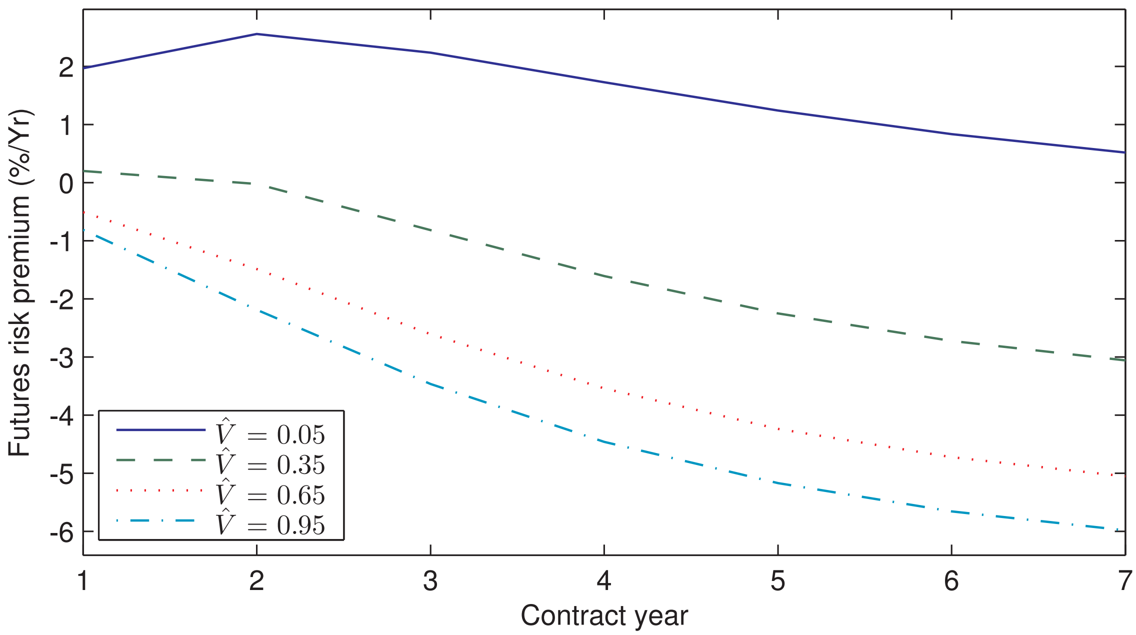
The probability "density" of $\hat{V}$ is plotted at increasing horizons, of 10, 25, 50, and 100 years, conditional on initial value $\hat{V}_0 = 0.05$. Although $\hat{V}$ has a discrete distribution conditional on $\hat{V}_0$, we plot a continuous analog to the probability mass function for ease of visualization. The resulting plot has two main features: (1) $\hat{V}$ exhibits an upward drift, such that values $\hat{V}_t > \hat{V}_0$ become very likely at longer horizons and (2) the probability mass becomes more dispersed, such that the range of probable values for $\hat{V}_t$ becomes much wider for larger t. Results are computed using Monte Carlo simulation with 10000 paths.

Figure 6: Prices and returns versus agent two promised utility, $\hat{V}$



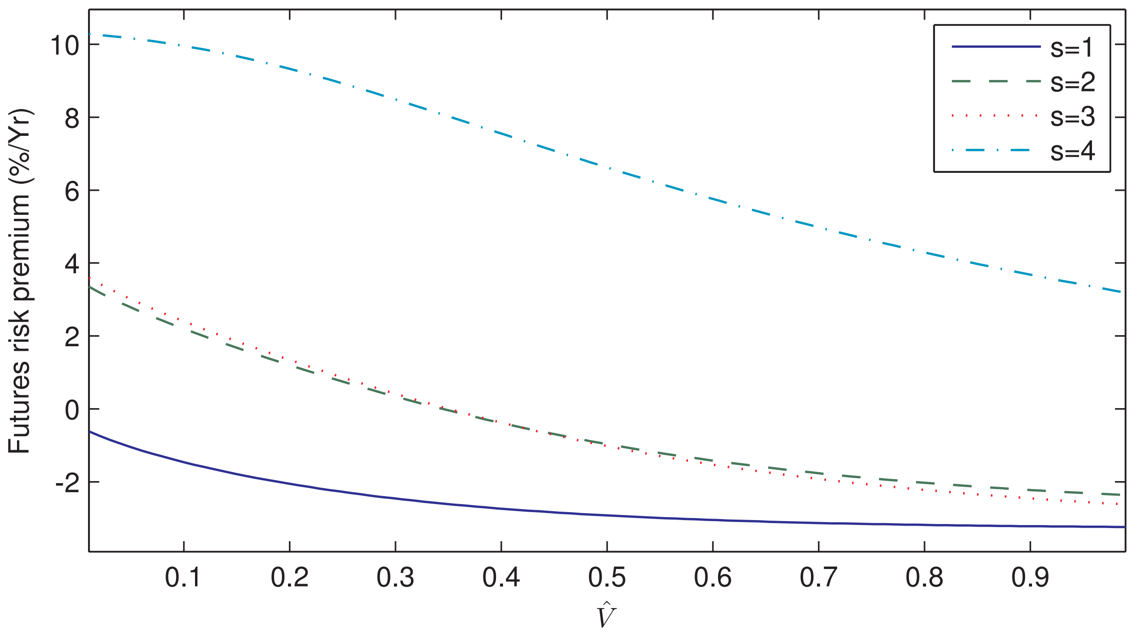
Summary of the model response to $\hat{V}$, showing the risk free rate, the spot price of oil, the equity premium, and the risk premium to a long position in the nearest futures contract. All values are expectations taken over the stationary distribution of growth states s. Rates and returns are annual.

Figure 7: Futures characteristics versus time to expiration



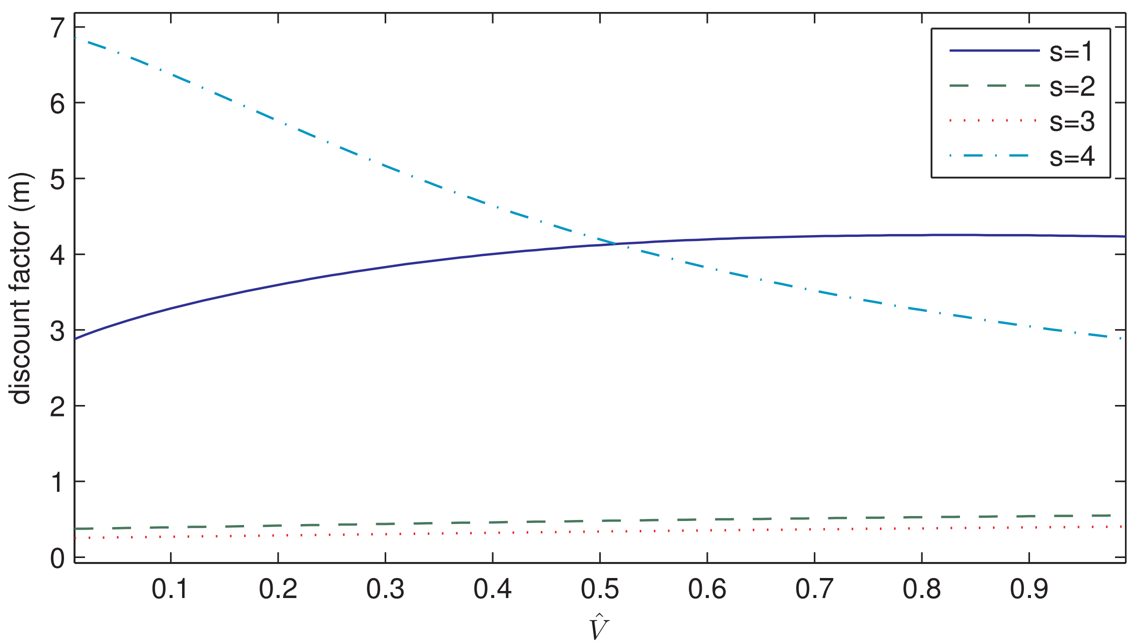
Annual model-implied risk premium to the long side of oil futures contracts with time to expiration of up to seven years. All values are expectations taken over the stationary distribution of growth states s.

Figure 8: Futures risk premium versus consumption share and growth state



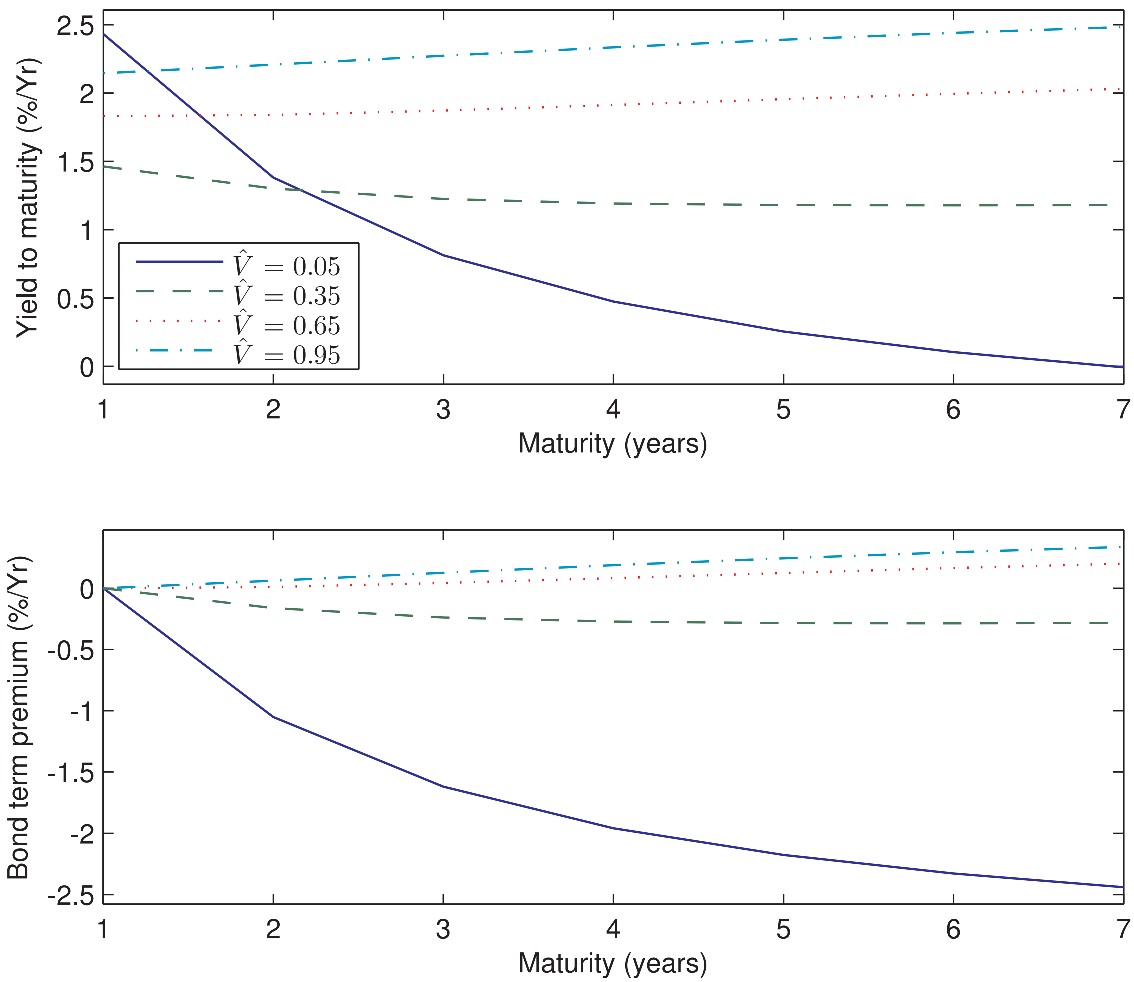
Annual model-implied risk premium to the long side of a 2-year oil futures contract, conditional on current Markov growth state s.

Figure 9: Discount factor



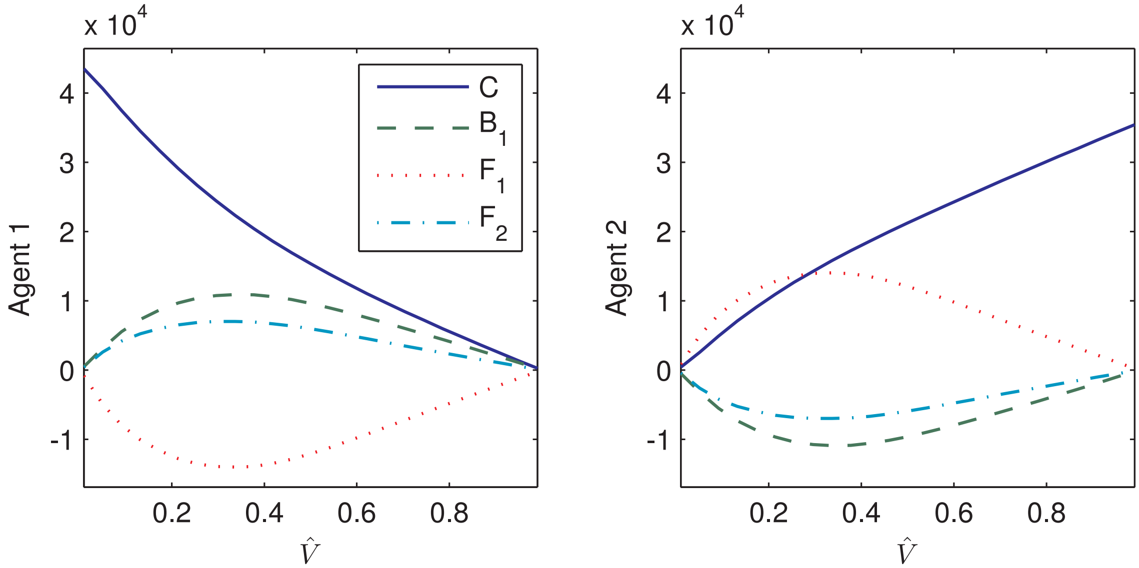
Conditional expectation of the stochastic discount factor, $E[m_{t+1}|\hat{V}_t, s_{t+1}]$. Each line conditions on a given realized state s_{t+1}. This illustrates which contingent claims are most valued in the market given agent two's share of aggregate wealth, for which $\hat{V}$ is a close proxy.

Figure 10: Zero-coupon bond characteristics versus maturity



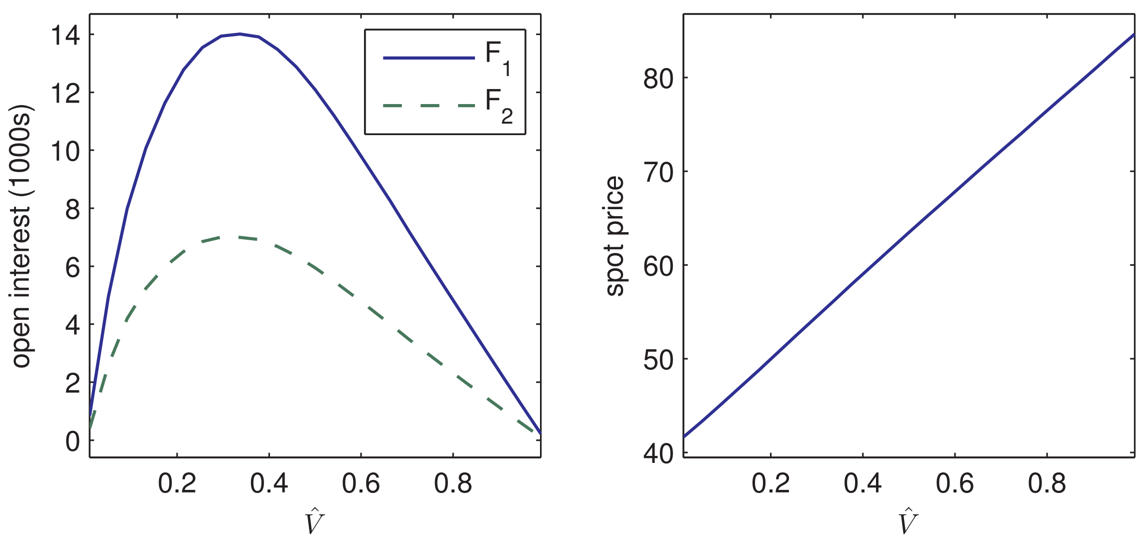
Annual model-implied yields to maturity and the model-implied term premium for zero coupon bonds, averaged over the stationary distribution of growth states.

Figure 11: Portfolios



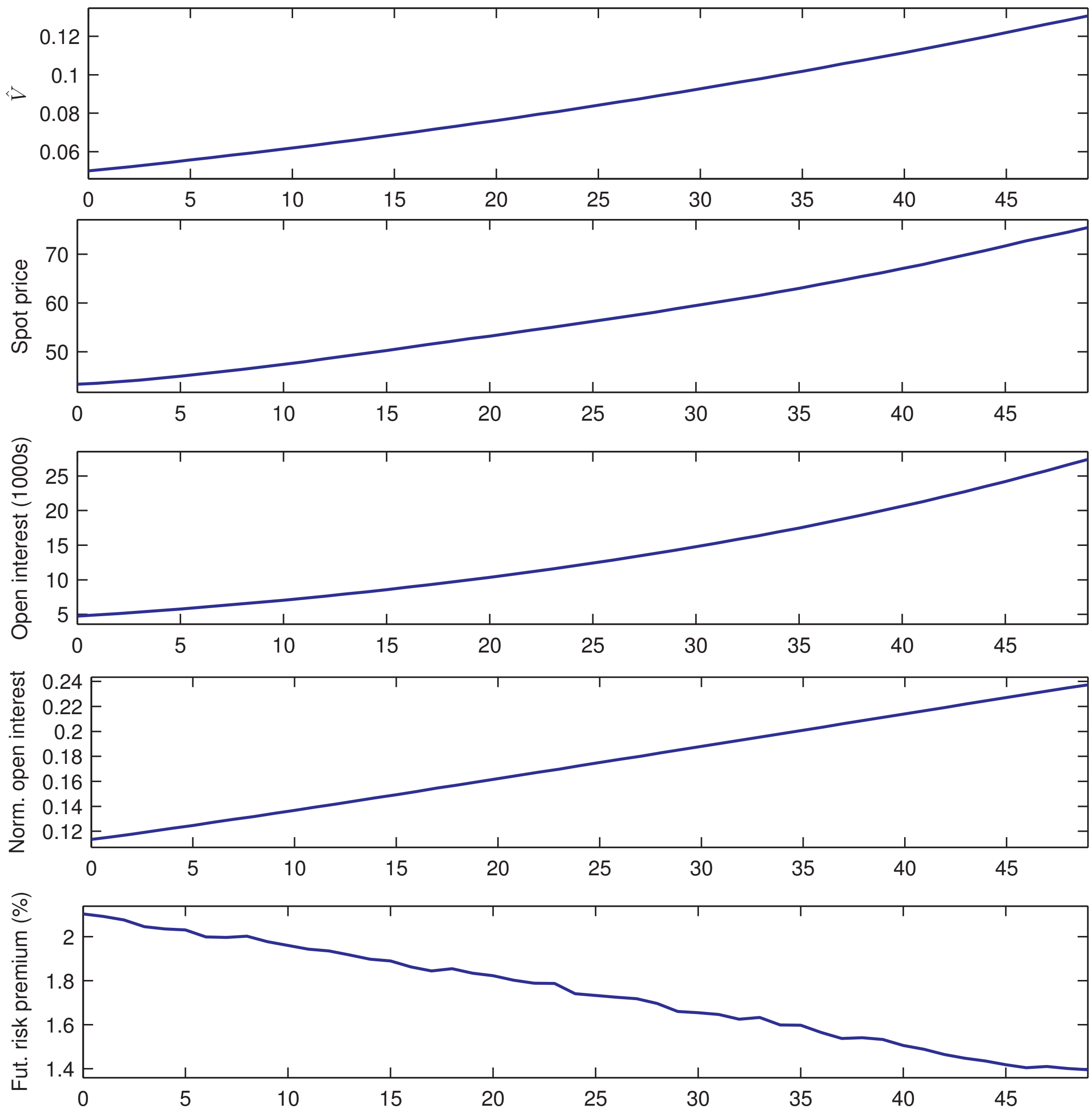
Portfolios for each agent, in terms of numeraire value of investment in each asset, versus $\hat{V}$. Plots are averages over growth states using the stationary distribution.

Figure 12: Open interest in oil futures and spot prices



The left panel shows model-implied open interest in 1 and 2 year oil futures contracts, expressed as the numeraire value of the contracts in 1000s, versus agent two promised utility $\hat{V}$. The right panel shows spot price of oil versus $\hat{V}$. Results are averaged over the stationary distribution of growth states s.

Figure 13: Evolution over time



Average path of the model economy over a 50-year period, conditional initial $\hat{V}_0 = 0.05$. The initial growth state is selected according to the stationary distribution. From top to bottom, the panels show $\hat{V}$ (indicative of agent two's wealth share), the oil spot price, open interest on 1-year futures, open interest normalized by aggregate wealth, and the risk premium on one-year oil futures. Results are computed using Monte Carlo simulation with 10000 paths.