A Model of Anomaly Discovery*

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Abstract

This paper analyzes the consequences of the discovery of anomalies. We show that consistent with existing evidence, the discovery of an anomaly reduces its magnitude and makes it more correlated with other existing anomalies. One new prediction is that the discovery of an anomaly reduces the correlation between the two extreme portfolios formed from the corresponding portfolio sorting for that anomaly. We empirically test this prediction for value, size, and momentum anomalies, and find strong evidence consistent with this prediction. Our model also sheds light on how to distinguish between risk- and mispricing-based anomalies.

Keywords: Anomaly, Arbitrage, Discovery, Arbitrageur-based asset pricing.

JEL Classifications: G11, G23.
1 Introduction

A significant portion of the asset-pricing literature has been devoted to discovering “anomalies,” empirical patterns that appear inconsistent with existing asset-pricing models. What are the consequences of discoveries on future asset prices and investor welfare? The answer to these questions critically depends on the interpretation of those anomalies. One main approach to understanding anomalies has been trying to offer rational risk-based explanations of the anomalies. Roughly speaking, this literature tries to explain why the assets with higher average returns are indeed more risky and that risk is not captured by existing models.

While this is an important approach to understanding anomalies, we argue that it abstracts away from the discovery aspect. By definition, a discovery should inform at least some investors of the phenomenon for the first time, unless the “discoverer” turns out to be the last person to learn the phenomenon. In a traditional model, however, a discovery does not inform anyone of the phenomenon, as everyone in the model understands the risk. Hence, the “discovery” has no asset-pricing effect.

However, it seems natural to expect discoveries to have significant effects on asset prices, since, over the years, discoveries in academia have been having increasingly important influences on the asset management industry.¹ In this paper, we explicitly model the discovery aspect and analyze its effects on asset prices, both theoretically and empirically.

We solve a model with two assets (assets 1 and 2) that have the same fundamentals (i.e., same distribution for future cash flows). However, investors find asset 1 more risky because their endowment is correlated with asset 1’s cash flow, but not with asset 2’s. Consequently, in equilibrium, asset 1 has a lower price and a higher expected future return than asset 2. We call this return pattern an “anomaly,” which is risk-based since it is caused by investors’ risk

¹Let’s take Dimensional Fund Advisors as an example. According to its website, as of June 30, 2014, it is managing $378 billion. Academic research appears to have a deep influence on its operations, as its website states: “Working closely with leading financial academics, we identify new ideas that may benefit investors.”
When this anomaly is discovered, some agents, who we call “arbitrageurs,” become aware of the return pattern and start exploiting it. We construct the equilibrium without these arbitrageurs, which we call a “pre-discovery equilibrium,” and the equilibrium with these arbitrageurs, which we call a “post-discovery equilibrium.” The discovery effect is captured by the difference between the pre- and post-discovery equilibria.²

Our model has the following implications. First, the discovery reduces the size of the anomaly. This follows directly once we recognize that the discovery brings in arbitrageurs. Let us use the value premium as an example. It has been proposed that value stocks are risky because they are more exposed to aggregate financial distress risk. Arbitrageurs, however, may not find value stocks risky, perhaps because they are wealthy and are less exposed to economy-wide financial distress. They are happy to exploit this anomaly, and consequently reduce its magnitude.

Second, the discovery makes the anomaly return (i.e., the return from a long position in asset 1 and a short position in asset 2) more correlated with the returns from other existing anomalies. This is due to the wealth effect when arbitrageurs exploit both existing anomalies and the newly discovered one. Suppose the return from the existing anomalies is unexpectedly high one period, thus increasing arbitrageurs’ wealth. Everything else being equal, the arbitrageurs will allocate more investment to all their opportunities, including the new anomaly. This higher investment pushes up the price of asset 1 and pushes down the price of asset 2, leading to a high return from the new anomaly this period. Similarly, a low return from the existing anomalies leads to a low return from the newly discovered one. Hence, the wealth effect increases the correlation between the new anomaly return and the returns from the existing anomalies.

Third, the discovery reduces the correlation coefficient between the returns of assets 1 and 2, and this effect is stronger when arbitrageurs’ wealth is more volatile. The intuition is as

²The traditional approach can be viewed as a special case, in which arbitrageurs’ wealth is zero. That is, everyone knew the anomaly all along. In this case, the pre- and post-discovery equilibria coincide and the discovery has no effect on asset prices.
follows. After the discovery, arbitrageurs have a long-short position in assets 1 and 2, as well as investments in other opportunities. Suppose the arbitrageurs’ wealth increases, say, due to a high return from their other investments, or fund flows from their investors. They will buy more of asset 1 and sell more of asset 2. This increases asset 1’s return but decreases asset 2’s. Similarly, when arbitrageurs’ wealth decreases, they will unwind some of their long-short positions, i.e., selling asset 1 and buying asset 2, which decreases asset 1’s return but increases asset 2’s. In both cases, arbitrageurs’ wealth shocks push the returns of the two assets to opposite directions, reducing their correlation. Naturally, this intuition also suggests that the effect is stronger when arbitrageurs’ wealth is more volatile.

The first two predictions are consistent with existing empirical evidence. For example, McLean and Pontiff (2013) analyzes the post-discovery performances of 82 anomalies, and finds that post-discovery anomaly returns decay by 44% on average. They also find that after the discovery, the new anomaly return becomes more correlated with the returns from existing anomalies.

The third prediction is new to the literature, and so we empirically test it. According to McLean and Pontiff (2013), there are more than 100 anomalies documented in the academic literature. However, one would expect that for a discovery to have a first-order effect on underlying stock prices, the anomaly needs to be widely known and exploited by a large number of arbitrageurs. Hence, we choose the three arguably most widely known anomalies: value, momentum, and size anomalies.

Our goal is to test, for each anomaly, whether the correlation between deciles 1 and 10, formed from the corresponding portfolio sort, decreases after the discovery of the anomaly. Essentially, the “discovery time” is the time when many arbitrageurs start exploiting the anomaly. One might suspect that it may take some time after arbitrageurs become aware of an anomaly for them to be convinced and start exploiting it. Moreover, the first publication might not be the one that generates most attention. For example, the discovery of size and value anomalies are usually credited to Banz (1981) and Basu (1983), respectively. However, one might suspect that Fama
and French (1992) has generated more attention on these two anomalies. Hence, we choose 1981 and 1992 as possible discovery time for the size anomaly, and use 1983 and 1992 as possible discovery time for the value anomaly. Finally, the discovery time for the momentum anomaly is chosen to be 1993 (Jegadeesh and Titman (1993)).

For each anomaly, we use a 5-year rolling window to estimate the correlation coefficient between the monthly excess returns of deciles 1 and 10 during 1927–2013. The estimates are plotted in Panels A–C of Figure 1. For momentum in Panel A, this correlation fluctuates between 0.6 and 0.8 in the first 6 decades. In the early 1990s, however, it dropped significantly to 0.4. This drop coincides with the publication of the most influential study on momentum (Jegadeesh and Titman (1993)) and the rise of the hedge fund industry. We also see similar large drops in this correlation for size and value anomalies in Panels B and C. Interestingly, although these two anomalies were documented in the early 1980s, the correlations did not drop sharply until 1992. This coincides with Fama and French’s influential work that narrows down a number of anomalies to size and value anomalies.

To formally test if the correlation between deciles 1 and 10 decreases after discovery, we control for its potential time trend by normalizing it with the correlation between deciles 5 and 6. The idea is that arbitrageurs are likely to take larger long-short positions in deciles 1 and 10 than in deciles 5 and 6. Hence, the correlation between deciles 5 and 6 should have a weaker discovery effect, but share the common time trend with the correlation between deciles 1 and 10. Hence, in our main tests, we use the correlation ratio: the correlation between deciles 1 and 10 divided by the correlation between deciles 5 and 6.

For each anomaly, we regress the correlation ratio on a dummy variable which takes the value of 0 before the discovery of the anomaly and 1 afterwards. For momentum, with the discovery time of 1993, the coefficient for the discovery dummy is $-0.13$, with a t-statistic of $-3.97$. This suggests that the post-discovery correlation ratio decreases by 0.13. For size and value anomalies, if we use the publication time of the original studies (Banz (1981) and Basu (1983)) as the discovery
time, the coefficient for the dummy is either insignificant or marginally significantly negative. In contrast, if we use the publication time for Fama and French (1992) as the discovery time, the coefficient estimates become twice as big for size and 50% larger for value. These results are consistent with the visual observations from the plots in Figure 1. There was no immediate decrease in the correlation coefficient after the original studies (Banz (1981) and Basu (1983)). Perhaps due to the wider influence of Fama and French (1992), and/or the rapid growth of the hedge fund industry, the correlations decreased significantly in the early 1990s.

Our most compelling evidence is the direct link between the correlation and arbitrageurs’ wealth. Our model predicts that the correlation is decreasing in the volatility of arbitrageurs’ wealth. We use the volatility of the aggregate asset under management (AUM) by hedge funds as a proxy for the volatility of arbitrageurs’ wealth. The implicit assumption is that the volatility of the aggregate wealth of hedge funds is positively correlated with the volatility of the total wealth of all arbitrageurs. We compute the volatility of the percentage changes in the aggregate AUM from Trading Advisor Selection System (TASS). Then, for each anomaly, we regress the correlation ratio on this hedge fund wealth volatility. Consistent with our model prediction, the coefficient for the wealth volatility is highly significant for all three anomalies: the coefficient estimate is $-6.78$ ($t = -2.23$), $-37.31$ ($t = -6.25$), and $-17.62$ ($t = -6.74$), for momentum, size, and value, respectively.

To contrast with the above risk-based anomaly, we also analyze a version of our model where the anomaly is due to mispricing. Specifically, we modify the previous model so that investors do not have the hedging demand in asset 1, but mistakenly believe that asset 1’s future cash flow is lower than asset 2’s. Consequently, the anomaly is not due to risk, but because of investors’ misperception. Our analysis shows that the discovery of this mispricing-based anomaly has exactly the same three predictions discussed above. As a result, analyzing asset prices cannot help distinguish between risk- and mispricing-based anomalies.

What is the solution then? One possibility is to examine investors’ portfolios instead, since
risk- and mispricing-based anomalies have starkly different implications for investors’ portfolios. For the value anomaly, for example, one can check if the investors who underweight value stocks appear to be those whose labor income or other assets are indeed more exposed to the proposed risk in the explanation. While this approach is demanding on the required dataset, in this big data era, with more and more micro-level datasets becoming available, this approach might not be a fantasy, either.

Our model is closely related to the analysis of arbitrageurs’ risk-bearing capacity (e.g., Dow and Gorton (1994), Shleifer and Vishny (1997), Xiong (2001) and Kyle and Xiong (2001)). More broadly, our paper belongs to the literature that explores the role of arbitrageurs in asset pricing (e.g., Gromb and Vayanos (2002), Liu and Longstaff (2004), Basak and Croitoru (2006), Brunnermeier and Pedersen (2009), Kondor (2009), He and Krishnamurthy (2013), and Kondor and Vayanos (2013)). These studies focus on the impact of arbitrageurs in contagion, risk sharing, liquidity, portfolio choice, and so on, while our paper focuses on the effect of discovery. While existing models of anomalies (e.g., Li, Livdan, and Zhang (2009) and the references therein) abstract away from the discovery aspect, we take it seriously and formally analyze its consequences. This discovery aspect, according to Cochrane (1999), “is (so far) the least stressed in academic analysis. In my opinion, it may end up being the most important.”

The rest of the paper is as follows. Sections 2 presents a model of risk-based anomaly, Section 3 analyzes a mispricing-based anomaly. Section 4 tests the main new predictions, Section 5 concludes. The numerical algorithm and proofs are in the appendix.

2 A model of the discovery of a risk-based anomaly

We consider a two-period model, with time $t = 0, 1, 2$. Trading takes place at $t = 0, 1$, and consumption occurs at $t = 2$. There is one risk-free asset, and its interest rate is normalized to 0. There are two risky assets, asset 1 and asset 2, each of which is a claim to a single cash flow.
at \( t = 2 \). There is a continuum of identical investors, with a population size of one. At \( t = 0 \), investors are endowed with one unit of both assets and \( k \) dollars cash.

The cash flows from assets 1 and 2 are independent and have the same \textit{ex ante} distribution. Specifically, for \( i = 1, 2 \), and \( t = 0, 1 \), we have

\[
D_{i,t+1} = D_{i,t} \times \mu_{i,t+1},
\]

where \( D_{i,0} = 1 \), and \( \mu_{i,t+1} \) are independent across \( i \) and \( t \), and have the same binary distribution

\[
\mu_{i,t+1} = \begin{cases} 
\mu + \sigma & \text{with probability } p, \\
\mu - \sigma & \text{with probability } 1 - p,
\end{cases}
\]

where \( \mu > \sigma > 0 \), and \( 0 < p < 1 \). Asset \( i \) is a claim to a cash flow \( D_{i,2} \) at time \( t = 2 \). Hence, the two cash flows (i.e., \( D_{1,2} \) and \( D_{2,2} \)) are independent from each other and have the same distribution at \( t = 0 \).

For \( i = 1, 2 \), and \( t = 0, 1, 2 \), we use \( P_{i,t} \) to denote the price of asset \( i \) at time \( t \), which will be determined endogenously in equilibrium. At \( t = 2 \), asset prices are pinned down by the final cash flow: \( P_{i,2} = D_{i,2} \). Let us denote the gross return of asset \( i \) at time \( t \) as

\[
r_{i,t} = \frac{P_{i,t}}{P_{i,t-1}},
\]

\subsection*{2.1 Risk-based anomaly}

In the literature, a phenomenon is labeled as an “anomaly” if the observed cross-sectional variation in average stock returns cannot be explained by existing standard models. For example, the evidence in Basu (1983) suggests that value stocks have higher average returns than growth stocks, and the return difference cannot be explained by the CAPM. Hence, this phenomenon is viewed as an anomaly. Many studies propose risk-based explanations of the value anomaly: Value stocks offer low returns during “bad times” and therefore demand a risk premium.

To capture the general notion of this risk-based explanation, we assume that investors have a hedging demand in one of the assets. Hence, in the equilibrium that we will characterize shortly,
the prices of the two assets differ at \( t = 0 \), although their payoffs at \( t = 2 \) have the same \( \text{ex ante} \) distribution.

Specifically, investors are endowed with a nontradable asset (e.g., labor income), which is a claim to a cash flow \( \rho D_{1,2} \) at \( t = 2 \), with \( \rho > 0 \). That is, this endowment is perfectly correlated with the payoff from asset 1. Denote investors’ wealth, excluding their nontradable endowment, at time \( t \) as \( W_t \) for \( t = 0, 1, 2 \). If investors allocate a fraction \( \theta_{i,t} \) of \( W_t \) to asset \( i \) at time \( t \), for \( i = 1, 2 \) and \( t = 0, 1 \), their wealth dynamic is given by

\[
W_{t+1} = W_t \left[ \sum_{i \in \{1,2\}} \theta_{i,t} r_{i,t+1} + \left( 1 - \sum_{i \in \{1,2\}} \theta_{i,t} \right) \right],
\]

for \( t = 0, 1 \), with \( W_0 = k + P_{1,0} + P_{2,0} \). Investors’ objective is to choose \( \theta_{i,t} \), for \( i = 1, 2 \), and \( t = 0, 1 \), to

\[
\max_{\theta_{i,t}} E_0 [\log (W_2 + \rho D_{1,2})],
\]

subject to (3).

In a reduced form, the above formulation captures the essence of risk-based interpretations of anomalies. In the value premium example, investors find value stocks risky and so reduce their demand for those stocks. Similarly, investors in our model find asset 1 risky because its return is correlated with their endowment.

### 2.2 Anomaly discovery

Traditional risk-based explanations of anomalies abstract away from the \textit{discovery aspect}. Let us use the value premium as an example. By definition, the discovery of the value premium in Basu (1983) should make at least \textit{some} investors aware of the return pattern for the first time, unless one believes Basu was actually the last person to find out the return pattern. In traditional risk-based models of value premium, however, \textit{all} investors knew the value premium even before the discovery in Basu (1983). That is, this traditional approach does not take into account the effect of discovery.
Our paper focuses exactly on this discovery aspect. In our model, a discovery makes some agents aware of the phenomenon for the first time. For convenience, we call those agents “arbitrageurs,” to highlight their difference from the previously-described investors whose objective is given by (4).

There is a continuum of identical arbitrageurs, with a population size of one. Their aggregate wealth at \( t = 0 \) is \( W_{a,0} \geq 0 \) dollars in cash. They have access to an investment opportunity, which presumably exploits other existing anomalies (say, e.g., currency carry trade). This opportunity is not available to the investors described earlier, perhaps because those investors do not have the expertise to analyze and implement the strategy. We call this existing anomaly “asset \( e \),” and assume its gross return at \( t = 1, 2 \) is

\[
    r_{e,t} = \begin{cases} 
    \mu_e + \sigma_e, & \text{with probability } p_e, \\
    \mu_e - \sigma_e, & \text{with probability } 1 - p_e, 
    \end{cases}
\]

where \( \mu_e > \sigma_e > 0 \), and \( 0 < p_e < 1 \). Moreover, \( r_{e,t} \) is assumed to be independent from \( D_{i,t} \). That is, the fundamentals of assets 1 and 2 are independent from the existing anomaly—asset \( e \).

To analyze the discovery effect, we compare the equilibria in the following two economies. In the first (pre-discovery) economy, arbitrageurs are not aware of the anomaly (i.e., assets 1 and 2 have the same fundamentals but different prices at \( t = 0 \)). Hence, they invest in asset \( e \), but not in assets 1 or 2. In the second (post-discovery) economy, arbitrageurs become aware of the anomaly and start exploiting it, as well as investing in the existing anomaly—asset \( e \). To capture this, we assume that arbitrageurs take a long-short strategy in the two assets so that they can exploit the anomaly and stay “market neutral.”

Specifically, we use \( \theta_{i,t}^a \) to denote the fraction of arbitrageurs’ wealth that is invested in asset \( i \in \{1, 2\} \) at time \( t \in \{0, 1\} \). A market-neutral

\[3\]This assumption is made so that the arbitrageurs focus on exploiting the anomaly. Alternatively, we can simply assume that after the discovery, the arbitrageurs become aware of the existence of assets 1 and 2. Under this alternative assumption, however, arbitrageurs will not only take a long-short position in the two assets, but also start investing in both assets. The latter will simply push up the prices of both assets. We are not interested in analyzing this latter effect. Moreover, in the value premium example, for instance, it seems more natural to think that, after the discovery of the value premium, hedge funds start buying value stocks and shorting growth stocks, rather than hedge funds becoming aware of the existence of both value and growth stocks and starting to buy both of them.
strategy is such that
\[ \theta_{1,t}^a + \theta_{2,t}^a = 0. \] (5)

Let us use \( \theta_{e,t}^a \) to denote the fraction of arbitrageurs’ wealth that is invested in asset \( e \) at time \( t = 0, 1 \). Then, arbitrageurs’ wealth dynamic is given by
\[
W_{t+1}^a = W_t^a \left[ \sum_{i \in \{1,2,e\}} \theta_{i,t}^a r_{i,t+1} + \left( 1 - \sum_{i \in \{1,2,e\}} \theta_{i,t}^a \right) \right],
\] (6)
for \( t = 0, 1 \). Their objective is to choose \( \theta_{i,t}^a \) for \( i = 1, 2, e \), to
\[
\max_{\theta_{i,t}^a} E_0 \left[ \log \left( W_2^a \right) \right],
\] (7)
subject to (5) and (6).

In the pre-discovery case, arbitrageurs are on the sidelines and have no impact on the markets for assets 1 and 2. Hence, the equilibrium can be defined as follows. The pre-discovery competitive equilibrium is defined as asset prices \( P_{i,t} \) for \( i = 1, 2 \), and \( t = 0, 1 \) and the investors’ portfolios \( \theta_{i,t} \) for \( t = 0, 1 \) and \( i = 1, 2 \), such that investors’ portfolios optimize (4), and markets clear, i.e., for \( i = 1, 2 \) and \( t = 0, 1 \),
\[ W_t \theta_{i,t} = P_{i,t}. \] (8)

Similarly, the post-discovery competitive equilibrium is defined as asset prices \( P_{i,t} \) for \( i = 1, 2 \), and \( t = 0, 1 \) and the portfolios of investors and arbitrageurs \( \theta_{i,t} \) for \( t = 0, 1 \) and \( i = 1, 2 \), and \( \theta_{a,t} \) for \( t = 0, 1, i = 1, 2, e \), such that investors’ portfolios optimize (4), arbitrageurs’ portfolios optimize (7), and markets clear, i.e., for \( i = 1, 2 \) and \( t = 0, 1 \),
\[ W_t \theta_{i,t} + W_t^a \theta_{a,t} = P_{i,t}. \] (9)

The implicit assumption is that the arbitrageurs do not have any hedging demand in asset 1 or 2. Moreover, after the discovery, they know that the cause of the anomaly is investors’ hedging demand. These are simplification assumptions. What is necessary is that the arbitrageurs have
less hedging demand in asset 1 than investors. Even if the arbitrageurs do not know the true cause of the anomaly, they will still invest in it and the main implications in this alternative model remain similar to those in our current setup.\footnote{See Brennan and Xia (2001) for an analysis of this intuition in the portfolio choice context.}

### 2.3 Equilibrium

**Proposition 1 (Pre-discovery)** The pre-discovery equilibrium prices $P_{i,t}$ and portfolio choices $\theta_{i,t}$ can be characterized by equation (8) and

$$E_t \left[ \frac{r_{i,t+1} - 1}{W_{t+1} + \rho P_{1,t+1}} \right] = 0, \text{ for } i = 1, 2, \ t = 1, 2.$$  

Moreover, in this equilibrium, we have $P_{1,0} < P_{2,0}$.

The above proposition illustrates the anomaly: Although both assets have the same fundamentals ex ante, they have different prices and hence different future expected returns. Due to their nontradable asset endowment, investors find asset 1 more risky than asset 2, leading to a lower price for asset 1.

This is of course a reduced-form formulation of a risk-based anomaly. While traditional risk-based models focus on the detailed analysis of the exact mechanism through which the hedging demand arises, they assume away the discovery aspect since all investors know the return pattern all along. In contrast, we are not interested in the details of the hedging demand, but focus on the analysis of the consequences of the discovery. The following proposition characterizes the post-discovery equilibrium.

**Proposition 2 (Post-discovery)** The post-discovery equilibrium prices $P_{i,t}$ and portfolio choices $\theta_{i,t}$, for $i = 1, 2$, $t = 1, 2$, and $\theta_{i,1}^e$, for $i = 1, 2, e$, can be characterized by equations (5), (9), and...
for $i = 1, 2$, $t = 1, 2$,

\[
E_t \left[ \frac{r_{i,t+1} - 1}{W_{t+1} + \rho P_{t+1}} \right] = 0,
\]

\[
E_t \left[ \frac{r_{1,t+1} - r_{2,t+1}}{W^a_{t+1}} \right] = 0,
\]

\[
E_t \left[ \frac{r_{e,t+1} - 1}{W^a_{t+1}} \right] = 0.
\]

Intuitively, since arbitrageurs are not exposed to the endowment risk investors have, they find the anomaly an attractive investment opportunity, and buy asset 1 and short asset 2. For convenience, we call the return from this long-short portfolio, $r_{1,1} - r_{2,1}$, the “anomaly return.”

To analyze the discovery effect, we will compare the post-discovery equilibrium in Proposition 2 with the pre-discovery equilibrium in Proposition 1.\textsuperscript{5} In particular, following the algorithm in Appendix A, we solve both equilibria numerically. The baseline parameter values are summarized in Table 1. In the following numerical analysis, we vary only one parameter at a time to examine the effects of the discovery. We have also repeated our numerical analyses for other parameter values and none of the following qualitative results are specific to the chosen parameters.

Figure 2 illustrates the effects of discovery on the expected anomaly returns. The dashed line represents the size of the anomaly (i.e., the expected anomaly return $E_0[r_{1,1} - r_{2,1}]$) before the discovery. Since arbitrageurs have no influence on the markets for assets 1 and 2 before the discovery, the dashed line is flat: The expected anomaly return is around 5.5% regardless of the arbitrageurs’ wealth.

After the discovery, arbitrageurs start exploiting the opportunity, reducing the expected anomaly return. As shown by the solid line in Panel A, the post-discovery expected anomaly

\textsuperscript{5}The equation system in Proposition 2 is highly nonlinear and we have not been able to establish the existence and uniqueness of their solutions. However, we have always been able to solve the equation system numerically, and the solution appears to be unique. One might be somewhat surprised that the simple two-period structure in our model does not allow for a closed-form solution. In fact, the wealth effect in our model has similar complexity as that in the continuous-time model in Xiong (2001), which also heavily relies on numerical analysis. As noted in Gromb and Vayanos (2002), a two-period model of arbitrageurs and investors with a wealth effect is not as tractable as its appearance suggests (page 381). In a recent study, Konder and Vayanos (2013) gain more tractability by simplifying investors’ decisions.
return is lower than that in the pre-discovery case (i.e., the solid line is below the dashed line). In the case $W^a_0 = 2$, for example, the discovery reduces the expected anomaly return from 5.5% to 5%.

The plot also shows that the effect of the discovery is stronger when arbitrageurs have more wealth. For example, in the case $W^a_0 = 5$, the discovery reduces the expected anomaly return from 5.5% to 4%. The discovery effect disappears when $W^a_0 = 0$. One can think of this $W^a_0 = 0$ case as representing the traditional modeling approach: The discovery in this case does not change the set of investors who are aware of the anomaly, since all investors knew about the anomaly all along.

Panels B and C demonstrate how arbitrageurs’ existing investment opportunity (i.e., asset $e$) affects the discovery effect. If arbitrageurs’ existing strategy is more attractive (i.e., $\mu_e$ is higher, or $\sigma_e$ is lower), upon the discovery of a new anomaly, they allocate less capital to exploit it and so the expected return of the new anomaly drops less. As shown in Panels B and C, the expected new anomaly return is increasing in $\mu_e$ and decreasing in $\sigma_e$.

### 2.4 Correlation among anomaly returns

By the construction of our model, before the discovery, the anomaly return $r_{1,1} - r_{2,1}$ is independent of the return of the existing anomaly $r_{e,1}$. How does the discovery affect the correlation between $r_{1,1} - r_{2,1}$ and $r_{e,1}$?

Intuitively, after the discovery of an anomaly, arbitrageurs start exploiting it, in addition to their investment in the existing anomaly, asset $e$. This creates a correlation through the wealth effect. Suppose the return from asset $e$ is unexpectedly high one period. This increases the wealth of these arbitrageurs. Everything else being equal, they will allocate more investment to the newly discovered anomaly. This higher investment pushes up the price of asset 1 and pushes down the price of asset 2, leading to a high anomaly return $r_{1,1} - r_{2,1}$. Similarly, an unexpectedly low return
from asset $e$ leads to a low anomaly return. That is, the wealth effect increases the correlation between the newly discovered anomaly return and the return from the existing anomaly.

The above intuition is illustrated in Figure 3. Panel A plots the correlation coefficient between $r_{1,1} - r_{2,1}$ and $r_{e,1}$. Before the discovery, as illustrated by the dashed line, the correlation is 0. In contrast, the post-discovery correlation, shown by the solid line, is positive. The only exception is the case $W_0 = 0$, where the correlation is zero, the same as in the pre-discovery case. Again, one can view this special case as the traditional approach that abstracts away from discovery: The discovery in this case does not influence the investor base.

This discovery effect (i.e., the change in the correlation across the pre- and post-discovery cases) is initially increasing in the size of the arbitrage capital $W_0$, and is not monotonic. This is because arbitrageurs have two effects on the correlation. The first is the above-mentioned wealth effect, which increases the correlation. The second is that arbitrage capital reduces the effect of hedging demand on the prices of the two assets. Hence, when the arbitrage capital increases, the long-short return $r_{1,1} - r_{2,1}$ is driven less by the hedging demand, but more by the fundamentals of the two assets, which are independent of the existing anomaly. This reduces the correlation between $r_{1,1} - r_{2,1}$ and $r_{e,1}$. When the size of the arbitrage capital is sufficiently large, the second effect dominates and a further increase in the arbitrage capital reduces the correlation.

The above intuition is further illustrated in Panels B and C. In particular, the wealth effect is stronger when arbitrageurs have a larger exposure to asset $e$. When arbitrageurs have a larger position in asset $e$, their wealth is more sensitive to the realized return of $r_{e,1}$. Hence, the discovery has a stronger effect on the correlation between $r_{1,1} - r_{2,1}$ and $r_{e,1}$. In Panel B, for example, as the expected return from asset $e$ increases (i.e., a higher $\mu_e$), it leads to a higher correlation between $r_{1,1} - r_{2,1}$ and $r_{e,1}$. Similarly, in Panel C, as the volatility of asset $e$ increases (i.e., a higher $\sigma_e$), it leads to a weaker wealth effect and a lower correlation.
2.5 Correlation between assets 1 and 2

Our model shows that the discovery of an anomaly reduces the correlation coefficient between the returns of assets 1 and 2. The intuition is as follows. After the discovery, arbitrageurs long asset 1 and short asset 2 to exploit the anomaly. Now, suppose the arbitrageurs’ wealth increases, say, due to a high return from their investment in asset e. They will buy more of asset 1 and sell more of asset 2. This increases asset 1’s return but decreases asset 2’s return. Similarly, when arbitrageurs’ wealth decreases, they will unwind some of their positions in the long-short portfolio. That is, they will sell asset 1 and buy asset 2, decreasing asset 1’s return but increasing asset 2’s return. In both cases, arbitrageurs’ wealth shocks push the returns of the two assets to opposite directions, which reduces the correlation between the returns of assets 1 and 2.

This intuition is illustrated in Figure 4. The dashed line in Panel A is for the pre-discovery correlation between assets 1 and 2. Since arbitrageurs are on the sidelines before the discovery, their wealth level $W_0$ does not affect the correlation. Hence, the dashed line is flat. The post-discovery case is represented by the solid line. It is below the dashed line, suggesting that the discovery reduces the correlation between assets 1 and 2. It also shows that the larger the size of the arbitrage capital, the larger the reduction in the correlation.

The above intuition further suggests that the discovery effect is stronger when arbitrageurs’ wealth is more volatile. To illustrate this intuition, we plot the correlation between assets 1 and 2 against arbitrageurs’ wealth volatility $\sigma^a$. Specifically, we vary arbitrageurs’ wealth volatility by changing the volatility of asset e. Panel B shows that the more volatile the arbitrageurs’ wealth, the stronger the discovery effect — the lower the correlation between assets 1 and 2.

3 Mispricing-based anomaly

To compare mispricing- and risk-based anomalies, we now analyze a model in which the anomaly is caused by investors’ behavioral bias. Specifically, we modify the previous model by setting
\( \rho = 0 \), that is, there is no hedging demand. The fundamentals of the two assets are still given by (1) and (2). However, investors are biased about asset 1 and believe that for \( t = 0, 1 \),

\[
\mu_{1,t+1} = \begin{cases} 
\mu + \sigma & \text{with probability } p - b, \\
\mu - \sigma & \text{with probability } p + b,
\end{cases}
\]

(10)

where \( 0 \leq b < p \). That is, investors underestimate asset 1’s expected cash flow, and \( b \) measures the degree of the bias. In contrast, investors’ belief about asset 2 is correct.

Investors’ objective is to choose \( \theta_{i,t} \), for \( i = 1, 2 \), and \( t = 0, 1 \) to

\[
\max_{\theta_{i,t}} \mathbb{E}_0^\rho \left[ \log (W_2) \right],
\]

subject to (3), where \( \mathbb{E}_0^\rho [\cdot] \) indicates that the expectation is taken under the biased belief in (10). Arbitrageurs have correct beliefs, and their objective is given by (7), as in the previous section.

This formulation is meant to capture the essence of mispricing-based interpretations of anomalies. For instance, in the value premium example, Lakonishok, Shleifer, and Vishny (1994) argues that investors are overly enthusiastic about glamorous growth stocks and have a low demand for value stocks. Similarly, in our model, investors underestimate the payoff from asset 1 and so have a low demand.

Similar to the case for the risk-based anomaly, in the pre-discovery case, arbitrageurs have no impact on the markets for assets 1 and 2. The competitive equilibrium for this case is defined as asset prices \( P_{i,t} \) for \( i = 1, 2 \), and \( t = 0, 1 \) and the investors’ portfolios \( \theta_{i,t} \) for \( t = 0, 1 \) and \( i = 1, 2 \), such that investors’ portfolios optimize (11), and markets clear as in (8).

The post-discovery competitive equilibrium is defined as asset prices \( P_{i,t} \) for \( i = 1, 2 \), and \( t = 0, 1 \) and the portfolios of investors and arbitrageurs \( \theta_{i,t} \) for \( t = 0, 1 \) and \( i = 1, 2 \), and \( \theta^a_{i,t} \) for \( t = 0, 1, i = 1, 2, e \), such that investors’ portfolios optimize (11), arbitrageurs’ portfolios optimize (7), and markets clear as in (9).

What is implicitly assumed here is that the discovery does not affect investors’ bias \( b \). That is, the bias is systematic and deeply rooted. These naive investors do not adjust their behavior
after the discovery of the anomaly.

**Proposition 3** The pre-discovery equilibrium prices $P_{i,t}$ and portfolio choices, $\theta_{i,t}$ can be characterized by (8) and for $i = 1, 2$, $t = 1, 2$,

$$E_t \left[ \frac{r_{i,t+1} - 1}{W_{t+1}} \right] = 0. \quad (12)$$

The post-discovery equilibrium prices $P_{i,t}$ and portfolio choices $\theta_{i,t}$, for $i = 1, 2$, $t = 1, 2$, and $\theta_{i,1}^{a}$, for $i = 1, 2, e$, can be characterized by equations (5), (9), (12), and for $t = 1, 2$,

$$E_t \left[ \frac{r_{1,t+1} - r_{2,t+1}}{W_{t+1}^{a}} \right] = 0, \quad (13)$$

$$E_t \left[ \frac{r_{e,t+1} - 1}{W_{t+1}^{a}} \right] = 0. \quad (14)$$

Similar to the risk-based case in the previous section, investors have a lower demand for asset 1 than for asset 2. The only difference is the motivation. In the risk-based case, investors demand less of asset 1 for hedging, while in the mispricing case, the motivation is investors’ wrong belief that asset 1 has lower cash flows in the future. Can we distinguish a risk-based anomaly from a mispricing-based one by examining asset prices? We examine this in the next section.

### 3.1 Compare risk- and mispricing-based anomalies

We now compare the risk-based case (Propositions 1 and 2) with the mispricing-based case (Proposition 3). In particular, we set $\rho = 0$ and $b = 0.055$, and adopt all other parameters from Table 1. We choose this value for $b$ so that, before the discovery, the expected anomaly returns are the same across the risk-based case and the mispricing-based case. We now compare the post-discovery return dynamic across the two cases.

Panel A of Figure 5 shows that it is difficult to distinguish a risk-based anomaly from a mispricing-based one by examining the post-discovery performance. The solid and dashed lines represent the post-discovery expected anomaly return for the risk- and mispricing-based cases,
respectively. The pre-discovery expected anomaly return for both cases is flat at around 5.5% (we omitted this flat line in this plot). The plot shows that the discovery of an anomaly reduces its expected return regardless of whether the anomaly is caused by risk or mispricing. Moreover, both lines are downward sloping, implying that the more arbitrage capital ($W_a$), the stronger the effect. The two lines are also close to each other, suggesting that the magnitude of the reduction of the anomaly return is similar across the two cases. Panel B shows that, for both the risk- and mispricing-based cases, the discovery of an anomaly increases the correlation between its return and the existing anomaly return. Even the non-monotonic pattern is similar across the two cases.

A number of empirical studies have analyzed the out-of-sample performance of anomaly returns. For example, McLean and Pontiff (2013) analyzes the post-discovery performances of 82 characteristics that have been identified in published academic studies. One of the goals of these studies is to distinguish between risk-based and mispricing-based interpretations.\(^6\) Our analysis, however, shows that the discovery of an anomaly reduces its magnitude, regardless of whether the anomaly is caused by risk or mispricing in the first place. Therefore, the post-discovery decay of an anomaly does not necessarily imply that it was caused by mispricing.

Finally, Panel C shows that the discovery of the anomaly reduces the correlation between assets 1 and 2 for both risk- and mispricing-based cases. Moreover, this correlation is decreasing in arbitrageurs’ wealth level $W_a$ in both cases.

### 3.2 One possible solution

The above results highlight the difficulty of distinguishing between risk- and mispricing-based anomalies by examining asset prices.\(^7\) What is the solution then? We argue that it is more promising to analyze investors’ portfolios. The idea is that investors’ holdings might offer direct evidence on why they overweight one asset and underweight another.

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\(^6\)Another important goal of this literature is to examine whether the anomalies are due to data mining, which is not analyzed in our model.

\(^7\)This is parallel to the result in Brav and Heaton (2002), which emphasizes the difficulty in distinguishing a biased belief from a rational belief with structural uncertainty.
In a risk-based anomaly, investors recognize the fact that asset 1’s expected return is higher than asset 2’s, and so they have a higher total exposure (including nontradable endowment) to asset 1 than to asset 2, despite that they underweight asset 1 in the stock market. In a mispricing-based anomaly, however, investors underweight asset 1 because they mistakenly believe that it has a lower future payoff.

Therefore, investors’ portfolio holdings can help separate risk and mispricing-based anomalies. For example, Fama and French (1993, 1996) interpret the value premium as value stocks exposing investors to risks associated with economy-wide financial distress. To evaluate this risk-based explanation of the value premium, one can examine whether the investors who underweight value stocks are those who are more exposed to risk of financial distress (e.g., their labor income or other assets are more exposed to financial distress).

To be fair, while examining portfolio holdings is a direct approach to distinguishing between risk- and mispricing-based anomalies, it is also very demanding on the dataset. It requires detailed information on investors’ positions, including their nontradable assets. Nevertheless, it is hopeful that as more micro-level data on investors’ holdings become available, this test may eventually become feasible. For example, Betermier, Calvet, and Sodini (2014) has recently analyzed the characteristics of investors of value and growth stocks, and potentially shed light on why investors hold value or growth stocks.

### 3.3 Welfare

How does the discovery of an anomaly affect investor welfare? To address this question, we first need to clarify our welfare measures. For the risk-based case, we simply use investors’ expected utility at $t = 0$. For the mispricing-based case, we use “subjective welfare” to refer to investors’ subjective expected utility at $t = 0$, and use “objective welfare” to refer to investors’ utility evaluated under the objective belief at $t = 0$. 

Proposition 4  The discovery of a risk-based anomaly increases investors’ welfare. The discovery of a mispricing-based anomaly increases investors’ subjective welfare, but reduces their objective welfare.

In the case of a risk-based anomaly, arbitrageurs essentially offer better risk sharing to investors. Before the discovery, the endowment risk is narrowly shared among investors (i.e., arbitrageurs are not involved). After the discovery, this endowment risk is shared between investors and arbitrageurs: investors unload asset 1 to arbitrageurs to hedge against their endowment risk. Arbitrageurs’ trading makes the hedging cheaper. For the mispricing-based case, when arbitrageurs start exploiting the anomaly, naive investors think they are better off, since they can offload some of asset 1, which they are pessimistic about. That is, investors’ subjective expected utility increases after the discovery.

However, the discovery reduces naive investors’ objective welfare. For instance, suppose the value premium was caused by investors’ misperception that glamorous growth stocks would outperform. The discovery of this anomaly attracts arbitrageurs to buy value and sell growth stocks. Consequently, investors end up holding more growth stocks and less value stocks, and will suffer from worse performances in the future.

4 Empirical evidence

Our risk-based model in Section 2 and mispricing-based model in Section 3 share the following three main predictions: The discovery of an anomaly i) reduces its magnitude, ii) increases the correlation of the anomaly return with the returns of other existing anomalies, and iii) reduces the correlation between assets 1 and 2.

The first two predictions are consistent with existing empirical evidence. For example, McLean and Pontiff (2013) analyzes the post-discovery performances of 82 characteristics that have been identified in published academic studies, They find that post-discovery anomaly returns decay by
44% on average. In addition, they find that after the discovery, the return of the new anomaly becomes more correlated with the returns from other existing anomalies.

The third prediction is new to the literature. We now empirically examine whether the discovery of the anomaly reduces the correlation between assets 1 and 2. According to McLean and Pontiff (2013), there are more than 100 anomalies documented in the academic literature. However, it is natural to expect that for a discovery to have a first-order effect on underlying stock prices, the anomaly needs to be widely known and exploited by a large number of arbitrageurs. Hence, we choose the three arguably most widely known anomalies: value, momentum, and size anomalies.

Our goal is to examine, for each anomaly, whether the correlation coefficient between the excess returns of deciles 1 and 10 from the corresponding portfolio sort decreases after the discovery of the anomaly. It is not obvious how to choose the “discovery time” for each anomaly. The decision is necessarily subjective to some extent. For example, should we use the time of the publication of the first study of the anomaly in the academic literature? One might object to this choice because it is possible that practitioners have known and exploited the anomaly before the first publication in the academic literature. Moreover, the essence of the “discovery time” is the time when a large number of arbitrageurs start exploiting the anomaly. One might suspect that it may take some time after arbitrageurs become aware of an anomaly for them to be convinced and start exploiting it. Moreover, the first publication might not be the one that generates most attention. For example, the discovery of the size and value anomalies are usually credited to Banz (1981) and Basu (1983), respectively. However, one might suspect that Fama and French (1992) has generated more attention on these two anomalies.

With this grain of salt in mind, we use March 1993 as the discovery time for momentum anomalies.

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8Lou and Polk (2013) uses the high-frequency correlation among stocks to infer the size of arbitrage capital. The economic mechanism is quite different. In their setup, higher high-frequency correlations among stocks imply a larger arbitrage capital size. However, in our setup, higher low-frequency correlations between decile 1 and 10 portfolios imply a smaller arbitrage capital size.

We obtain the monthly decile portfolio returns and risk-free asset returns during 1927–2013 from Kenneth French’s website. Panel A of Figure 1 is the time series of the correlation coefficients of the returns from deciles 1 and 10 of momentum portfolios, estimated based on a rolling window of the previous 5 years’ data. In the first 6 decades, this correlation fluctuates between 0.6 and 0.8. In the early 1990s, however, it dropped significantly to 0.4. The significant decline in the correlation did not occur until the publication of the most influential study on momentum (Jegadeesh and Titman (1993)). Although we cannot infer causality, this change in correlation is certainly consistent with our model prediction. Similarly, Panels B and C are the correlation coefficients for size and value anomalies, respectively. Interestingly, although these two anomalies were documented in the early 1980s, the correlations started dropping sharply in 1992. This coincides with Fama and French’s influential work that singles out size and value among a number of anomalies. It is also interesting to note that both drops were accompanied by a concurrent surge of the hedge fund industry.

We formally test whether the correlations between anomaly portfolios decrease after discoveries. In particular, we normalize the correlations between deciles 1 and 10 by the correlations between deciles 5 and 6, in order to control for a potential time trend. The motivation is the following. To exploit the anomaly return, arbitrageurs are likely to take larger long-short positions in deciles 1 and 10 than in deciles 5 and 6. That is, the correlation between deciles 5 and 6 may share a common time trend with the correlation between deciles 1 and 10, but should be subject to a weaker discovery effect. For each anomaly, we compute the correlation ratio as the following

\[ X_t = \frac{\hat{\rho}_{1,10}^t}{\hat{\rho}_{5,6}^t}, \]  

where \( \hat{\rho}_{1,10}^t \) is the correlation coefficient between the monthly excess returns of portfolios 1 and 10 of the anomaly during the five years prior to month \( t \), and \( \hat{\rho}_{5,6}^t \) is similarly defined for portfolios.
Because $\rho_{5,6}^t$ is stable and close to 1, the ratios are very similar to the raw correlations in the upper plots, as we can see from Panels D–F of Figure 1.

Table 2 reports the summary statistics of the estimated correlation ratios. The first column of Panel A shows that, for momentum, the correlation ratio has a mean of 0.55, and standard deviation of 0.08. The correlation ratio is slightly higher (0.66 and 0.75) and more volatile (0.21 and 0.11) for size and value anomalies, respectively. For each of the three series, the mean and median are close to each other.

To test our model prediction, for each anomaly, we regress the ratio $X_t$ on Discovery, a dummy variable which takes the value of 0 before the discovery of the anomaly and 1 afterwards. For the momentum anomaly, for example, we use March 1993 (the publication time of Jegadeesh and Titman (1993)) as the discovery time. The first column of Table 3 shows that the coefficient for the discovery dummy is $-0.13$, with a $t$-statistic of $3.97$. This suggests that the post-discovery correlation ratio decreases by 0.13.

For the size anomaly, as discussed earlier, we analyze two potential discovery time, September 1981 and June 1992. If we use the former as the discovery time, as shown in the second column, the coefficient of the discovery dummy is negative. However, this estimate is statistically insignificant. Once we use June 1992 as the discovery time, however, the result gets stronger: The point estimate becomes twice as large, $-0.14$, with a $t$-statistic of $-1.78$. Similarly, for the value anomaly, the coefficient for the discovery dummy is $-0.09$, only marginally significant, if we use June 1983 as the discovery time (column 4). Once we use June 1992 as the discovery time, the point estimate becomes $-0.13$, with a $t$-statistic of $-1.96$. These results are consistent with the observation from the plots in Figure 1. There is no immediate decrease in the correlation coefficient after the original studies (Banz (1981) and Basu (1983)). Perhaps due to the wider influence of Fama and French (1992), and/or the rapid growth of the hedge fund industry, the correlations decrease significantly in the early 1990s.
The essential point of our model is that arbitrageurs’ trading activity reduces the correlation between the two portfolios. We now try to analyze more directly whether this correlation is indeed related to arbitrageurs’ activity. As shown in Panel B of Figure 4, the correlation is decreasing in \( \sigma^a \), the volatility of arbitrageurs’ wealth. So, we will test whether the correlation ratio is correlated with the volatility of arbitrageurs’ wealth.

It is certainly impossible to directly observe the aggregate wealth of all arbitrageurs. As a compromise, we will measure the wealth of one group of investors, who are often considered to be arbitrageurs in financial markets: hedge funds. The implicit assumption is that the volatility of the aggregate wealth of hedge funds is positively correlated with the volatility of the total wealth of all arbitrageurs.

We obtain monthly hedge fund returns and AUM from TASS during 1994—2012. Then, we compute the percentage change in AUM for each fund and aggregate them into the value-weighted average of percentage AUM change of all funds. For each month from 1999—2012, hedge fund wealth volatility, denoted by \( \text{wealth}_\text{vol} \), is calculated as the standard deviation of this aggregate AUM change during the previous 5 years. The summary statistics for \( \text{wealth}_\text{vol} \) are reported in the first column of Panel B in Table 2. This series has a mean of 0.032 and a standard deviation of 0.005. The median is 0.031, nearly identical to the mean.

Since \( \text{wealth}_\text{vol} \) directly corresponds to arbitrageurs’ wealth volatility \( \sigma^a \), we can test our model prediction by examining whether the correlation ratio \( X \) is indeed negatively related to \( \text{wealth}_\text{vol} \). The first column of each panel of Table 4 reports the regression results of the correlation ratio \( X \) on hedge funds’ wealth volatility, \( \text{wealth}_\text{vol} \). As a control variable, we also include \( VIX \), the implied volatility of S&P 500 index options on the Chicago Board of Exchange in the current month. Consistent with our model prediction, the coefficient for \( \text{wealth}_\text{vol} \) is highly significant for all three anomalies: The coefficient estimate is \(-6.78 \text{ (} t = -2.23 \text{)}, -37.31 \text{ (} t = -6.25 \text{)}, \) and \(-17.62 \text{ (} t = -6.74 \text{)}, \) for momentum, size, and value, respectively. \( VIX \) is considered to be negatively related to the amount of arbitrage capital in the market (e.g., Brun-
nermeier and Pedersen, 2009). Hence, a high VIX is correlated with less arbitrage capital and so a high correlation ratio X. In our estimates in Panels B and C, the coefficient for VIX is highly positively significant, although the estimate for momentum in Panel A is insignificant.

The AUM change can be decomposed into two components: the first is due to performances and the second is due to fund flows from investors. Similar to the construction of wealth_vol, we compute the volatility of the each component. For each month, we use the data from the previous 5 years to estimate the standard deviation of the monthly value-weighted hedge fund returns, ret_vol_t, and the standard deviation of the monthly percentage flows to the hedge fund industry, flow_vol_t. Panel B in Table 2 shows that flow_vol_t is slightly more volatile than ret_vol_t. The correlation coefficient between these two components is 0.51, as shown in Panel C. It also shows that both flow_vol_t and ret_vol_t are negatively correlated to the correlation ratio X for all three anomalies.

We then regress X_t on ret_vol_t, flow_vol_t and VIX_t for each anomaly. The results are reported in Table 4. The second and third columns of Panels A–C show that, if we only include one of flow_vol_t and ret_vol_t in the regressions, the coefficients for both are significantly negative, with one exception of flow_vol_t for momentum. Including both flow_vol_t and ret_vol_t in the regressions, we find that for momentum, the return-induced volatility ret_vol_t is more correlated with X, while for size and value, the fund-flow-induced volatility flow_vol_t is more correlated with X.

We also conducted a variety of robustness analyses and the main results remain similar throughout. First, we use the correlation estimate \( \rho_{1,10}^t \) (rather than the correlation ratio \( X_t \)) to conduct our analysis. Second, we use the average correlation of 1000 pairs of simulated random decile portfolios to replace \( \rho_{5,6}^t \) in our construction of X. Third, we use a 3-year rolling window (rather than the 5-year rolling window in the above analysis) to estimate the correlation ratios in equation (15). Forth, we use quintile portfolios to construct the correlation ratio: the correlation coefficient between quintiles 1 and 5 divided by the correlation coefficient between...
quintiles 2 and 4. Finally, we use the index return from Hedge Fund Research (rather than the one we constructed from TASS) for our regressions in Table 4.

5 Conclusion

We have analyzed a simple model of anomaly discovery. It shows that consistent with existing evidence, the discovery of an anomaly reduces its magnitude and makes it more correlated with other existing anomalies. Our model’s new prediction is that the discovery of an anomaly reduces the correlation between the two extreme portfolios formed from the corresponding portfolio sorting for that anomaly. Moreover, this effect is stronger when arbitrageurs’ wealth is more volatile. We empirically test these new predictions for value, size, and momentum anomalies, and find strong evidence consistent with our new predictions.

Our analysis also contrasts risk- with mispricing-based anomalies, and highlights the difficulty of distinguishing between them by examining asset prices. We argue that one solution is to examine investors’ portfolios instead. For value premium, for example, one can examine the exposures of those investors who underweight value stocks and overweight growth stocks. Do they choose growth stocks to hedge their risk? Are they more exposed to the risk proposed in the explanation? While this direct approach is demanding on the required dataset, in this big data era, it might not be a fantasy, either.
References


Appendix A. Numerical procedure

We follow the procedure described below to solve the model:

1. Take initial guesses for the total wealth for investors and arbitrageurs at \( t = 1 \): \( W_1 \) and \( W_a^1 \) for the eight states at date 1.

2. For each of the eight states, take \( W_1 \) and \( W_a^1 \) as given, solve for the portfolios \( \theta_{i,1} \) for \( i = 1, 2 \), and \( \theta_{a,1} \) for \( i = 1, 2, e \) and prices \( P_{1,1} \) and \( P_{2,1} \).

3. Take the prices \( P_{1,1} \) and \( P_{2,1} \) for the eight states in step 2 as given, solve for the \( t = 0 \) portfolios \( \theta_{i,0} \) for \( i = 1, 2 \), and \( \theta_{a,0} \) for \( i = 1, 2, e \) and prices \( P_{1,0} \) and \( P_{2,0} \).

4. Based on the portfolios in step 3 \( \theta_{i,0} \) for \( i = 1, 2 \), and \( \theta_{a,0} \) for \( i = 1, 2, e \) and the prices in steps 2 and 3 \( P_{1,0}, P_{2,0}, P_{1,1}, P_{2,1} \) for all eight states at \( t = 1 \), calculate the investors’ and arbitrageurs’ updated wealth, \( W_1 \) and \( W_a^1 \), in the eight cases at \( t = 1 \).

5. Repeat steps 2 to 4 until the wealth, portfolios, and prices converge, i.e., for each variable, the difference between two iterations is no greater than 0.00005.

Appendix B. Proofs

Proof of Propositions 1 and 2

Due to the logarithmic preference, the maximization problem (4) is equivalent to maximizing the log wealth growth for each period. Hence, investors’ first-order conditions are given by

\[
E_t \left[ \frac{r_{i,t+1} - 1}{W_{t+1} + \rho P_{1,t+1}} \right] = 0,
\]

for \( i = 1, 2, t = 0, 1 \). Similarly, the arbitrageurs’ optimization problem (7) can also be decomposed into a period-by-period optimization problem, and the first-order conditions are given by

\[
E_t \left[ \frac{r_{1,t+1} - r_{2,t+1}}{W_{t+1}^a} \right] = 0,
E_t \left[ \frac{r_{a,t+1} - 1}{W_{t+1}^a} \right] = 0.
\]

Combining the above first-order conditions with the market-clearing conditions, we can characterize the equilibria in Propositions 1 and 2.

We now prove \( P_{1,0} < P_{2,0} \) by contradiction. Suppose \( P_{1,0} \geq P_{2,0} \). Note that investors’ optimal portfolio in equilibrium is to hold one unit of both assets. Suppose an investor sells \( \epsilon \) unit of asset 1 and buys \( \epsilon \) unit of asset 2. Define his expected utility as

\[
U(\epsilon) \equiv E_0[\text{Log}(k + (1 + \rho - \epsilon)D_{1,2} + (1 + \epsilon)D_{2,2})].
\]
It is easy to see that $\frac{dU}{d\epsilon}|_{\epsilon=0} > 0$. That is, he can strictly improve his portfolio by selling $\epsilon$ unit of asset 1 and buying $\epsilon$ unit of asset 2. This leads to a contradiction.

**Proof of Propositions 3**

The first-order condition to the maximization problem (11) is given by (12). The first-order conditions for arbitrageurs are still given by (13) and (14). These optimality and market-clearing conditions lead to the results in the proposition.

**Proof of Proposition 4**

In both the risk-based and mispricing-based cases, investors have the option not to trade. The participation constraint implies that the investors’ expected utility cannot be lower than that in the pre-discovery case. Moreover, investors’ concave utility function and convex budget constraint imply that the investors’ optimization problem has a unique solution. It is easy to see that in the case of $W_a > 0$, the portfolio characterized in Proposition 2 is strictly different from the non-participation portfolio. Hence, the discovery strictly increases investor welfare. Similarly, in the mispricing-based case, the discovery strictly increases investors’ subjective welfare.

To analyze naive investors’ objective welfare, we note that naive investors’ portfolio in the economy in Section 3 can be decomposed into one unit in assets 1 and 2, and a position $x_t$ (for $t = 0, 1$) in the long-short strategy (long asset 1 and short asset 2). It is easy to show that naive investors’ objective welfare $E[\text{long}(W_2)]$ is concave in $x_0$ and $x_1$. In the pre-discovery case, $x_0 = x_1 = 0$. In the post-discovery case, however, $x_t$ is “further away” from the optimum point for maximizing $E[\text{log}(W_2)]$. For example, an naive investor’s choice is $x_0 < 0$ altough $\partial E[\text{log}(W_2)]/\partial x_0|_{x_0=0} > 0$. Therefore, an naive investor’s objective welfare is lower in the post-discovery case.
Table 1: Baseline Parameterizations

<table>
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<th>Parameter</th>
<th>$W_0^a$</th>
<th>$k$</th>
<th>$\rho$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$p$</th>
<th>$\mu_e$</th>
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This table reports the baseline parameter values in our numerical analysis.
Table 2: Summary Statistics

Panel A: Correlation Ratios (1932-2013)

<table>
<thead>
<tr>
<th></th>
<th>$X_{mom}$</th>
<th>$X_{size}$</th>
<th>$X_{value}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.55</td>
<td>0.66</td>
<td>0.75</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.08</td>
<td>0.21</td>
<td>0.11</td>
</tr>
<tr>
<td>Median</td>
<td>0.51</td>
<td>0.72</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Panel B: Volatility (1999-2012)

<table>
<thead>
<tr>
<th></th>
<th>$wealth_{vol}$</th>
<th>$ret_{vol}$</th>
<th>$flow_{vol}$</th>
<th>$VIX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.032</td>
<td>0.019</td>
<td>0.025</td>
<td>22.069</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.005</td>
<td>0.004</td>
<td>0.004</td>
<td>8.270</td>
</tr>
<tr>
<td>Median</td>
<td>0.031</td>
<td>0.019</td>
<td>0.024</td>
<td>21.205</td>
</tr>
</tbody>
</table>

Panel C: Correlation Matrix (1999-2013)

<table>
<thead>
<tr>
<th></th>
<th>$X_{mom}$</th>
<th>$X_{size}$</th>
<th>$X_{value}$</th>
<th>$wealth_{vol}$</th>
<th>$ret_{vol}$</th>
<th>$flow_{vol}$</th>
<th>$VIX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{mom}$</td>
<td>1.00</td>
<td>0.18</td>
<td>0.03</td>
<td>-0.42</td>
<td>-0.48</td>
<td>-0.19</td>
<td>-0.16</td>
</tr>
<tr>
<td>$X_{size}$</td>
<td>1.00</td>
<td>0.87</td>
<td>-0.78</td>
<td>-0.54</td>
<td>-0.86</td>
<td>0.07</td>
<td></td>
</tr>
<tr>
<td>$X_{value}$</td>
<td>1.00</td>
<td>-0.64</td>
<td>1.00</td>
<td>0.79</td>
<td>0.89</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$wealth_{vol}$</td>
<td>1.00</td>
<td>0.79</td>
<td>0.89</td>
<td>1.00</td>
<td>0.06</td>
<td>0.44</td>
<td></td>
</tr>
<tr>
<td>$ret_{vol}$</td>
<td>1.00</td>
<td>0.51</td>
<td>0.06</td>
<td>1.00</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$flow_{vol}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$VIX$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
</tr>
</tbody>
</table>

Panel A reports the mean, standard deviation, and median of the correlation ratio $X_t$, defined in (15). $X_{mom}$, $X_{size}$, and $X_{value}$ are for momentum, size and value anomalies, respectively. Panel B reports the mean, standard deviation, and median of $wealth_{vol}$, $ret_{vol}$, $flow_{vol}$ and $VIX$. $wealth_{vol}$ is the standard deviation of the monthly percentage changes in the total asset under management by all hedge funds. $ret_{vol}$ is the standard deviation of the monthly value weighted returns among all hedge funds reported in TASS. $flow_{vol}$ is the standard deviation of the monthly percentage flows to all hedge funds in TASS. These three variables are estimated based on a rolling window of the prior 60-month data. We only include funds that have been reporting to TASS for more than or equal to 2 months. $VIX$ is the monthly series of the implied volatility of S&P 500 index options on the Chicago Board of Exchange.
### Table 3: Pre- and Post-discovery Correlations (Jan 1932–Dec 2012)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Discovery</td>
<td>-0.13***</td>
<td>-0.07</td>
<td>-0.14*</td>
<td>-0.09*</td>
<td>-0.13**</td>
</tr>
<tr>
<td></td>
<td>(-3.97)</td>
<td>(-1.15)</td>
<td>(-1.78)</td>
<td>(-1.75)</td>
<td>(-1.96)</td>
</tr>
<tr>
<td>adj $R^2$</td>
<td>0.387</td>
<td>0.075</td>
<td>0.236</td>
<td>0.126</td>
<td>0.194</td>
</tr>
</tbody>
</table>

This table reports, for each anomaly, the results from regressions of the correlation ratio $X_t$, defined in (15), on the dummy variable $Discovery$, which is 0 before the discovery time and 1 afterwards. The discovery time is in the parenthesis at the top of each column. The $t$-statistics are reported in the parenthesis under each coefficient estimate, and are based on the standard errors with Newey-West adjustment with 60 lags. *, **, and *** indicate that the coefficients are statistically significant at 10%, 5%, and 1% level, respectively.
Table 4: Correlations and Hedge Funds (Jan 1999–Dec 2012)

Panel A: Momentum

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>wealth_vol</td>
<td>-6.78***</td>
<td>-2.23</td>
</tr>
<tr>
<td>ret_vol</td>
<td>-11.16**</td>
<td>-12.26*</td>
</tr>
<tr>
<td></td>
<td>-2.36</td>
<td>(-1.88)</td>
</tr>
<tr>
<td>flow_vol</td>
<td>-3.83</td>
<td>1.76</td>
</tr>
<tr>
<td></td>
<td>-0.70</td>
<td>(0.30)</td>
</tr>
<tr>
<td>VIX</td>
<td>-0.05</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>-0.54</td>
<td>(1.59)</td>
</tr>
<tr>
<td></td>
<td>(1.98)</td>
<td>(1.10)</td>
</tr>
<tr>
<td>adj_R^2</td>
<td>0.17</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Panel B: Size

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>wealth_vol</td>
<td>-37.31***</td>
<td>-6.25</td>
</tr>
<tr>
<td>ret_vol</td>
<td>-41.29***</td>
<td>-15.55**</td>
</tr>
<tr>
<td></td>
<td>-4.31</td>
<td>-2.18</td>
</tr>
<tr>
<td>flow_vol</td>
<td>-48.33***</td>
<td>-41.24***</td>
</tr>
<tr>
<td></td>
<td>-12.52</td>
<td>-8.95</td>
</tr>
<tr>
<td>VIX</td>
<td>0.71***</td>
<td>0.95***</td>
</tr>
<tr>
<td></td>
<td>6.37</td>
<td>5.03</td>
</tr>
<tr>
<td></td>
<td>(2.52)</td>
<td>(5.02)</td>
</tr>
<tr>
<td>adj_R^2</td>
<td>0.67</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Panel C: Value

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>wealth_vol</td>
<td>-17.62***</td>
<td>-6.74</td>
</tr>
<tr>
<td>ret_vol</td>
<td>-13.88***</td>
<td>1.83</td>
</tr>
<tr>
<td></td>
<td>-3.98</td>
<td>(1.23)</td>
</tr>
<tr>
<td>flow_vol</td>
<td>-24.32***</td>
<td>-25.16***</td>
</tr>
<tr>
<td></td>
<td>-22.08</td>
<td>-20.49</td>
</tr>
<tr>
<td>VIX</td>
<td>0.63***</td>
<td>0.64***</td>
</tr>
<tr>
<td></td>
<td>7.27</td>
<td>5.89</td>
</tr>
<tr>
<td></td>
<td>(6.99)</td>
<td>(6.87)</td>
</tr>
<tr>
<td>adj_R^2</td>
<td>0.62</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>0.76</td>
<td>0.76</td>
</tr>
</tbody>
</table>

This table reports, for each anomaly, the results of contemporaneous regressions of the correlation ratio $X_t$, defined in (15), on $wealth_{vol}$, $ret_{vol_t}$, $flow_{vol_t}$, and $VIX_t$. $wealth_{vol}$ is the standard deviation of the monthly percentage changes in the asset under management by all hedge funds during the prior 60 months, $ret_{vol_t}$, the standard deviation of the monthly hedge fund index returns during the prior 60 months, $flow_{vol_t}$, the standard deviation of the monthly flows in percentage to the all hedge funds during the prior 60 months, and $VIX_t$, the implied volatility of S&P 500 index options in the current month. Hedge funds data are from the TASS database. We only include funds that have been reporting to the database for more than or equal to 2 months. The $t$-statistics are reported in the parenthesis under each coefficient estimate, and are based on the standard errors with the Newey-West adjustment with 60 lags. *, **, and *** indicate that the coefficients are statistically significant at 10%, 5%, and 1% level, respectively.
Panels A–C plot the correlation coefficient between the excess returns of deciles 1 and 10 for momentum, size and value, respectively, Panel D–F plot the correlation ratio, defined in (15).
Panels A–C plot the expected anomaly return, $E[r_{1,1} - r_{2,1}]$, on arbitrageurs’ initial wealth $W_0^a$, asset $e$’s expected return $\mu_e$ and volatility $\sigma_e$, respectively. The parameter values are given by Table 1.
Panels A–C plot the correlation coefficient between the anomaly return and asset $e$’s return, $\text{Corr}(r_{1,1} - r_{2,1}, r_{e,1})$, on the arbitrageurs' initial wealth $W_0^a$, asset $e$’s expected return $\mu_e$, and its volatility $\sigma_e$, respectively. The parameter values are given by Table 1.
Panels A and B plot the correlation coefficient between assets 1 and 2, $Corr(r_{1,1}, r_{2,1})$, on the arbitrageurs' initial wealth $W_0^a$, and their wealth volatility $\sigma^a$, respectively. The parameter values are given by Table 1.
Figure 5: Comparison: Asset Prices

Panels A–C plot the expected anomaly return, \( E[r_{1,1} - r_{2,1}] \), its correlation with asset \( e \)'s return, \( Corr(r_{1,1} - r_{2,1}, r_{e,1}) \), and the correlation between assets 1 and 2, \( Corr(r_{1,1}, r_{2,1}) \), on the arbitrageurs' initial wealth \( W^a_0 \), respectively. The solid line is for the case of risk-based anomaly, and the dashed line the mispricing-based anomaly. Parameter values: \( b = 0.055 \), and other parameter values are given by Table 1.